CS224d: Deep NLP

Lecture 12: Midterm Review

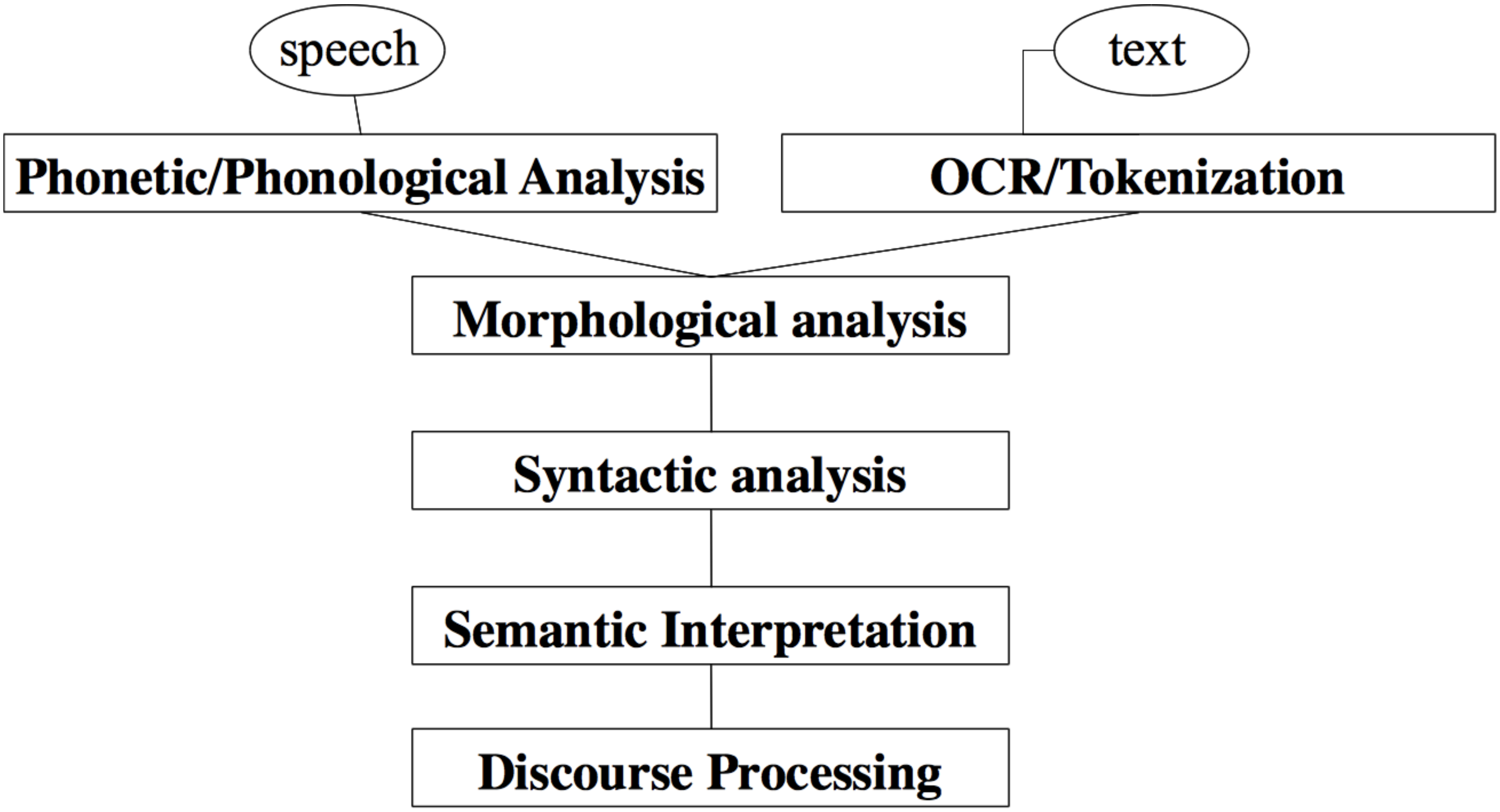
Richard Socher

richard@metamind.io

Overview Today – Mostly open for questions!

- Linguistic Background: Levels and tasks
- Word Vectors
- Backprop
- RNNs

Overview of linguistic levels



Tasks: NER

Named Entity Recognition

The acquisition will beef up Markham , Ontario-based Magna 's North American
U-LOCATION U-MISC U-ORGANIZATION B-MISC L-MISC

car and truck seating business , allowing it to better compete with Johnson
B-ORGANIZATION

Controls Inc and Lear Corp. Around 2,000 of Indonesia
I-ORGANIZATION L-ORGANIZATION B-ORGANIZATION L-ORGANIZATION U-LOCATION

's controversial Timor national car made by Kia Motor Corp of
U-MISC B-ORGANIZATION I-ORGANIZATION L-ORGANIZATION

South Korea arrived at Jakarta 's Tanjung Priok port on Thursday .
B-LOCATION L-LOCATION U-LOCATION B-LOCATION L-LOCATION

Tasks: POS

Part of Speech Tagging

Acting on a tip from spelunkers two years ago, scientists in South Africa discovered
NNP IN DT NN IN CD CD NNS JJ NNS IN NNP NNP VBD

what the cavers had only dimly glimpsed through a crack in a limestone wall deep in
WP DT NNS VBD RB RB -NONE- IN DT NN IN DT JJ NN RB IN

the Rising Star Cave: lots and lots of old bones. The remains covered the earthen floor
DT NNP NNP JJ NNS CC NNS IN JJ NNP DT VBZ VBN DT JJ NN

beyond the narrow opening. This was, the scientists concluded, a large, dark chamber
IN DT JJ NNP DT VBZ DT NNS -NONE- DT JJ JJ NN

for the dead of a previously unidentified species of the early human lineage — Homo
IN DT JJ IN DT RB JJ NNS IN DT JJ JJ NNP NN NN

naledi.
NNP

Tasks: Sentiment analysis

Sentiment Analysis

the best way to hope for any chance of enjoying this film is by lowering your expectations .

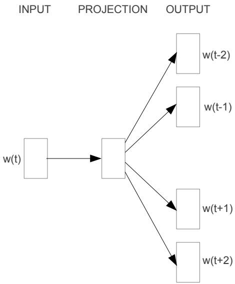
negative

Machine Translation

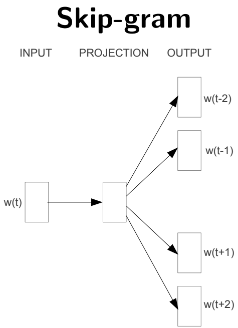
Original	Translation
I am a man of yesterday's culture.	ich bin ein Mann der gestrigen Kultur . Explain
I grew up on examples of artists who lived poor and died in poverty, refused money for the sake of painting.	ich wuchs an Beispielen von Künstlern , die arm lebten und in Armut starben . Explain
This is the culture I'm for.	das ist die Kultur , für die ich bin . Explain
There is such a thing: an official for culture.	es gibt so etwas : ein Beamter für die Kultur . Explain

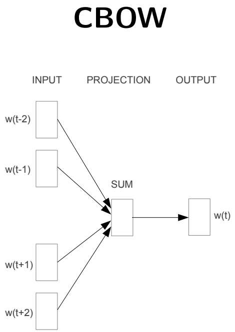
Skip-gram

- *Task:* given a center word, predict its context words
- For each word, we have an "input vector" v_w and an "output vector" v'_w



Skip-gram v.s. CBOW





	Skip-gram	CBOW
Task	Center word $\rightarrow$ Context	Context $\rightarrow$ Center word
r	v_{w_i}	$f(v_{w_{i-C}}, \cdots, v_{w_{i-1}}, v_{w_{i+1}}, \cdots, v_{w_{i+C}})$

All `word2vec` figures are from http://arxiv.org/pdf/1301.3781.pdf

word2vec as matrix factorization (conceptually)

- Matrix factorization

$$\begin{bmatrix} \quad M \quad \end{bmatrix}_{n\times n} \approx \begin{bmatrix} A^\top \end{bmatrix}_{n\times k} \begin{bmatrix} \quad B \quad \end{bmatrix}_{k\times n}$$

$$M_{ij} \approx a_i^\top b_j$$

- Imagine M is a matrix of counts for events co-occurring, but we only get to observe the co-occurrences one at a time. E.g.

$$M = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 0 & 2 \\ 1 & 3 & 0 \end{bmatrix}$$

but we only see

(1,1), (2,3), (3,2), (2,3), (1,3), ...

word2vec as matrix factorization (conceptually)

$$M_{ij} \approx a_i^\top b_j$$

- Whenever we see a pair (i, j) co-occur, we try to increasing $a_i^\top b_j$
- We also try to make all the other inner-products smaller to account for pairs never observed (or unobserved yet), by decreasing $a_{\neg i}^\top b_j$ and $a_i^\top b_{\neg j}$
- Remember from the lecture that the word co-occurrence matrix usually captures the semantic meaning of a word? For `word2vec` models, roughly speaking, M is the windowed word co-occurrence matrix, A is the output vector matrix, and B is the input vector matrix.
- Why not just use one set of vectors? It's equivalent to $A = B$ in our formulation here, but less constraints is usually easier for optimization.

GloVe v.s. `word2vec`

	Fast training	Efficient usage of statistics	Quality affected by size of corpora	Captures complex patterns
Direct prediction (`word2vec`)	Scales with size of corpus	No	No*	Yes
GloVe	Yes	Yes	No	Yes

* Skip-gram and CBOW are qualitatively different when it comes to smaller corpora

Overview

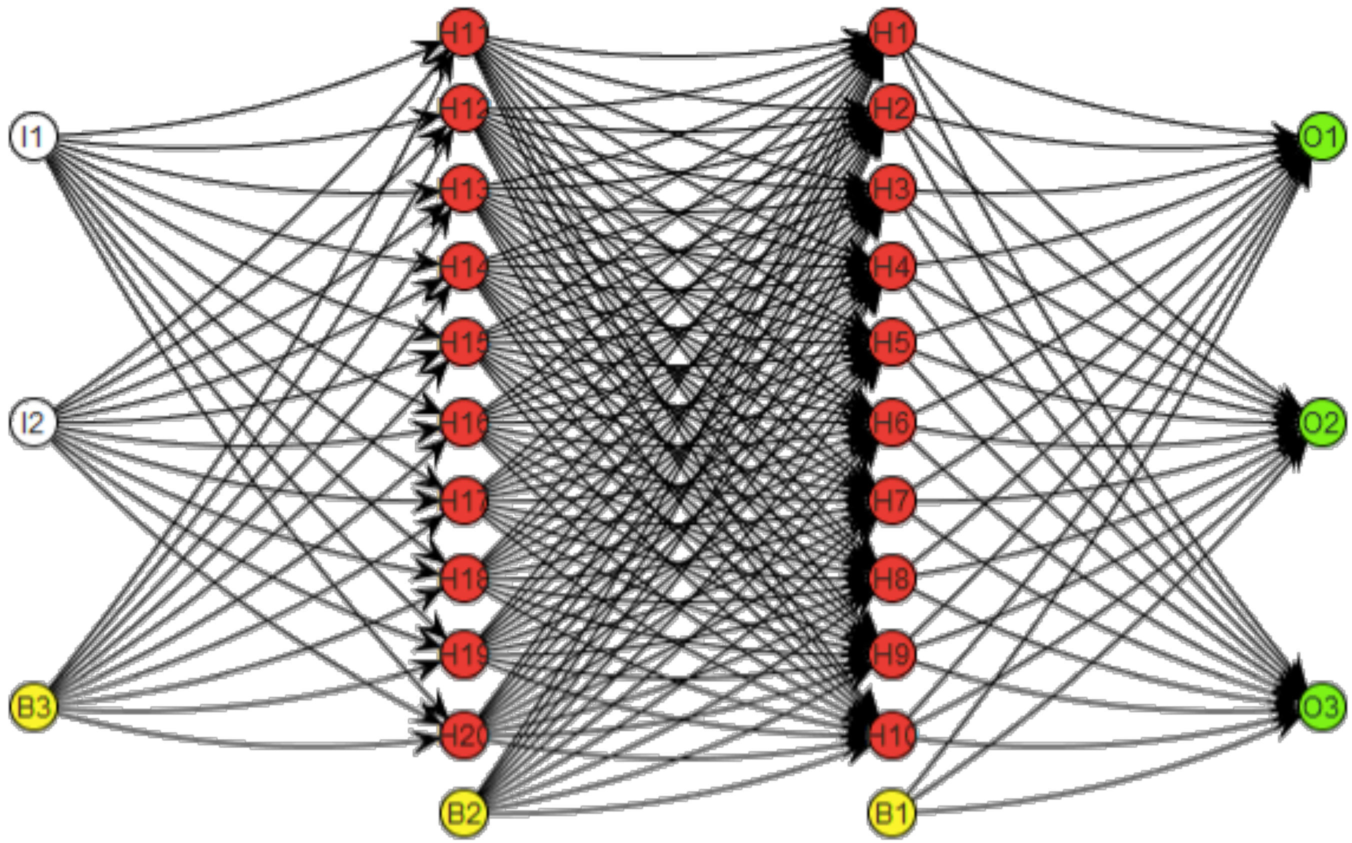
- Neural Network Example
- Terminology
- Example 1:
  - Forward Pass
  - Backpropagation Using Chain Rule
  - **What is delta? From Chain Rule to Modular Error Flow**
- Example 2:
  - Forward Pass
  - Backpropagation

Neural Networks

- One of many different types of non-linear classifiers (i.e. leads to non-linear decision boundaries)

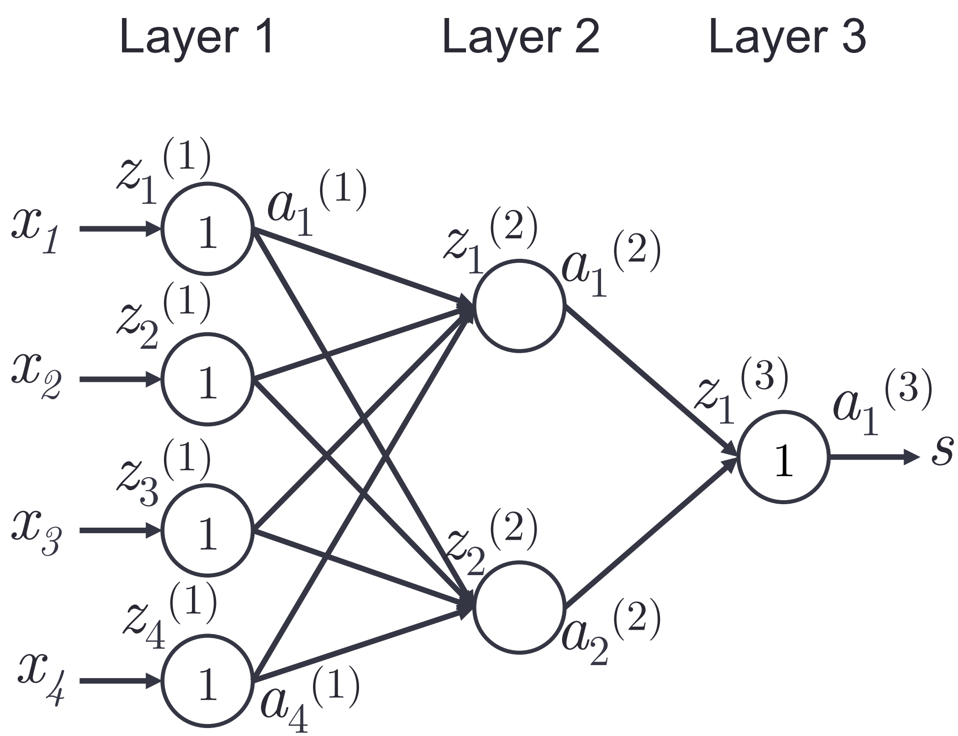
- Most common design involves the stacking of affine transformations followed by point-wise (element-wise) non-linearity

An example of a neural network



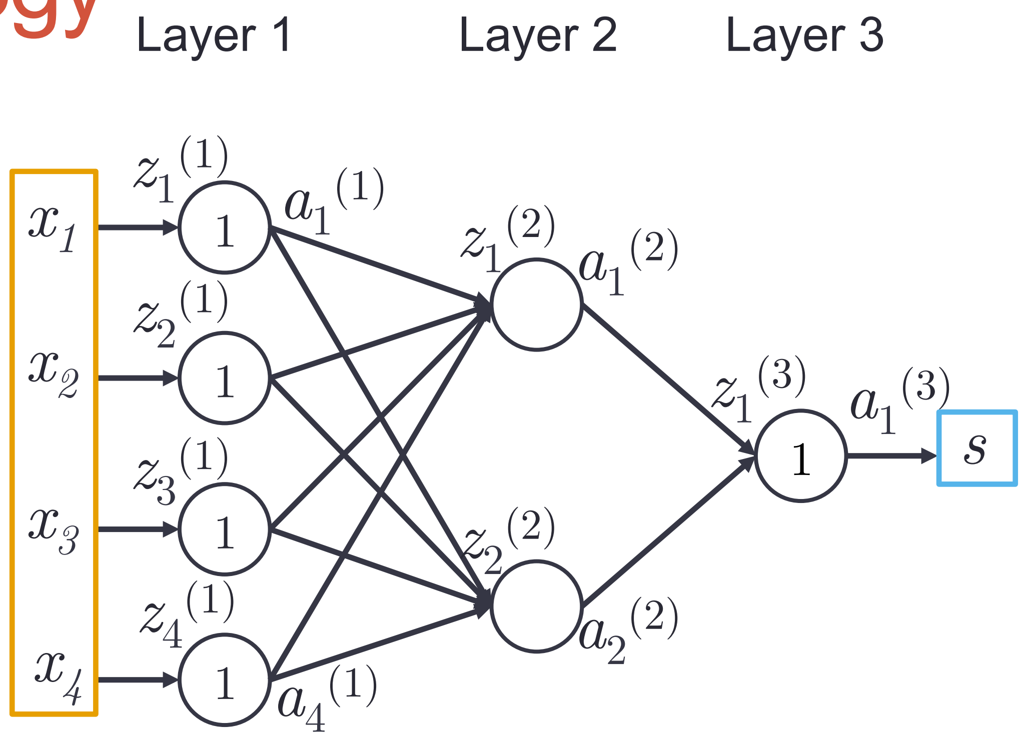
- This is a 4 layer neural network.
- 2 hidden-layer neural network.
- 2-10-10-3 neural network (complete architecture defn.)

Our first example



- This is a 3 layer neural network
- 1 hidden-layer neural network

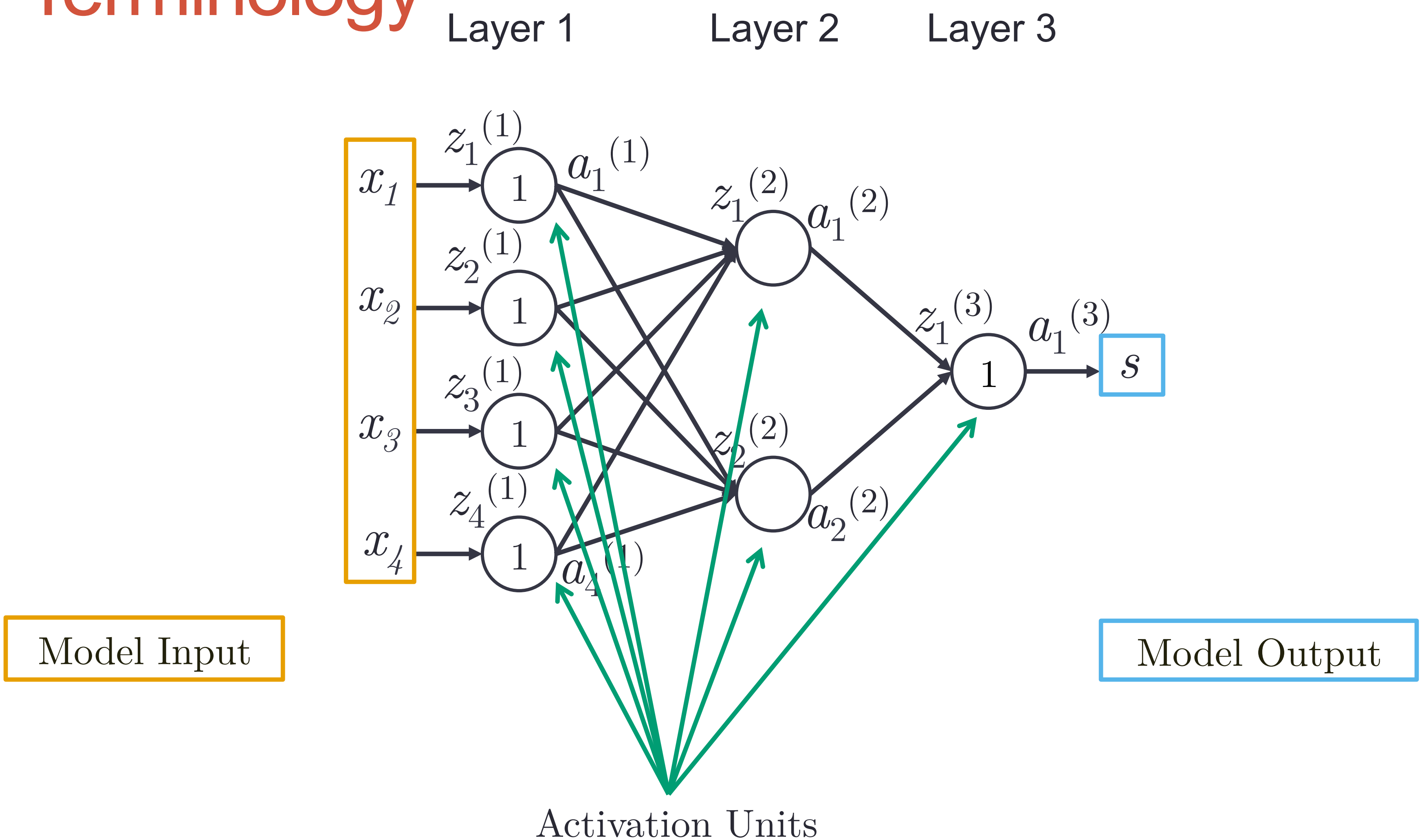
Our first example: Terminology



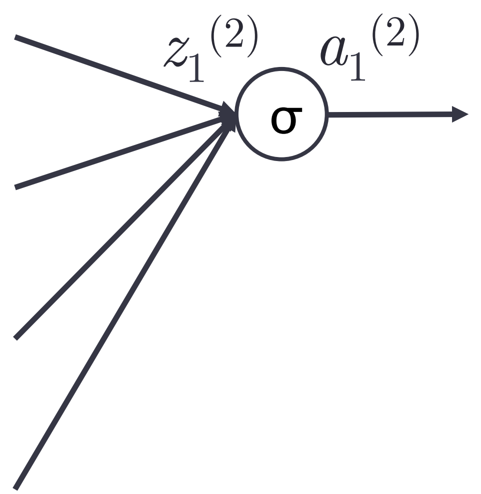
Model Input

Model Output

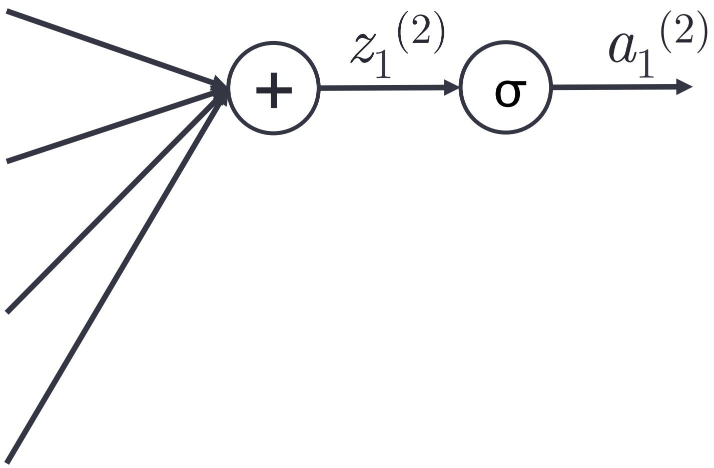
Our first example: Terminology



Our first example: Activation Unit Terminology



We draw this



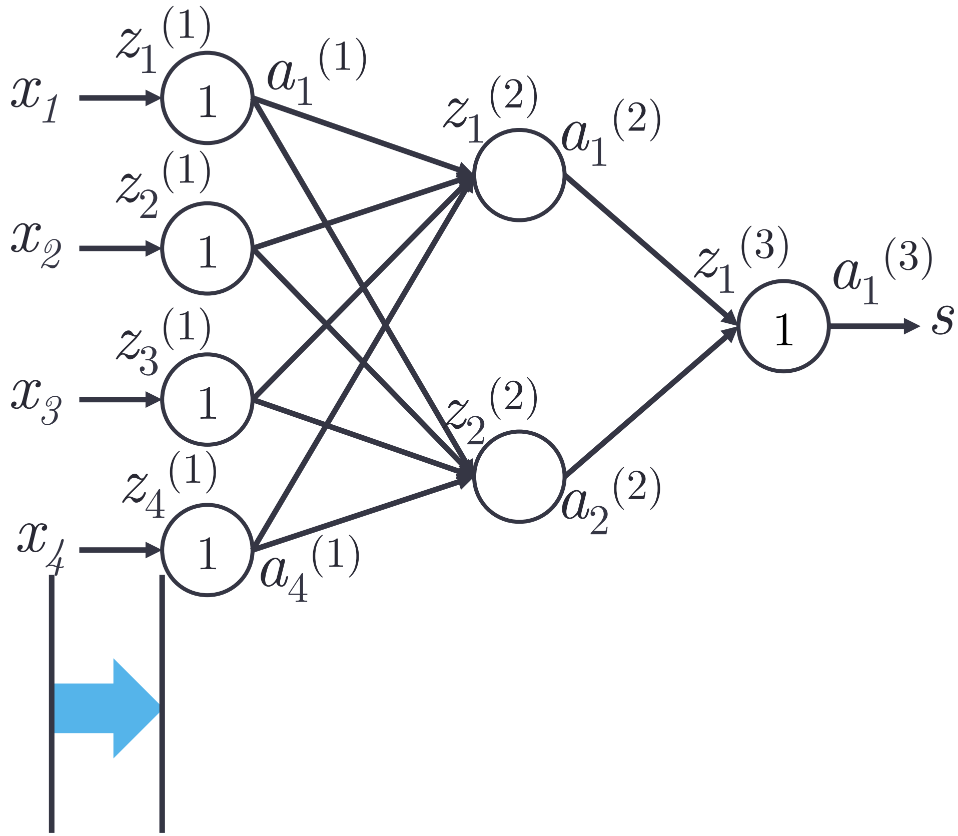
This is actually what's going on

$$z_1^{(2)} = W_{11}^{(1)} a_1^{(1)} + W_{12}^{(1)} a_2^{(1)} + W_{13}^{(1)} a_3^{(1)} + W_{14}^{(1)} a_4^{(1)}$$

$a_1^{(2)}$ is the 1^{st} activation unit of layer 2

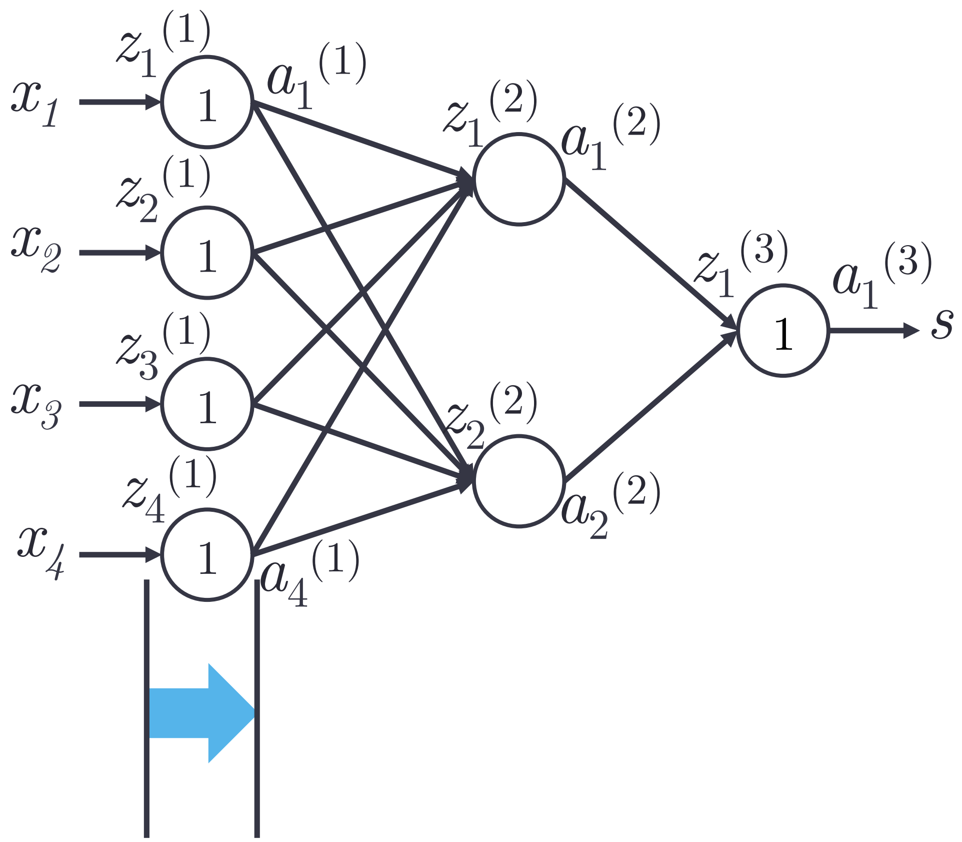
$$a_1^{(2)} = \sigma(z_1^{(2)})$$

Our first example: Forward Pass



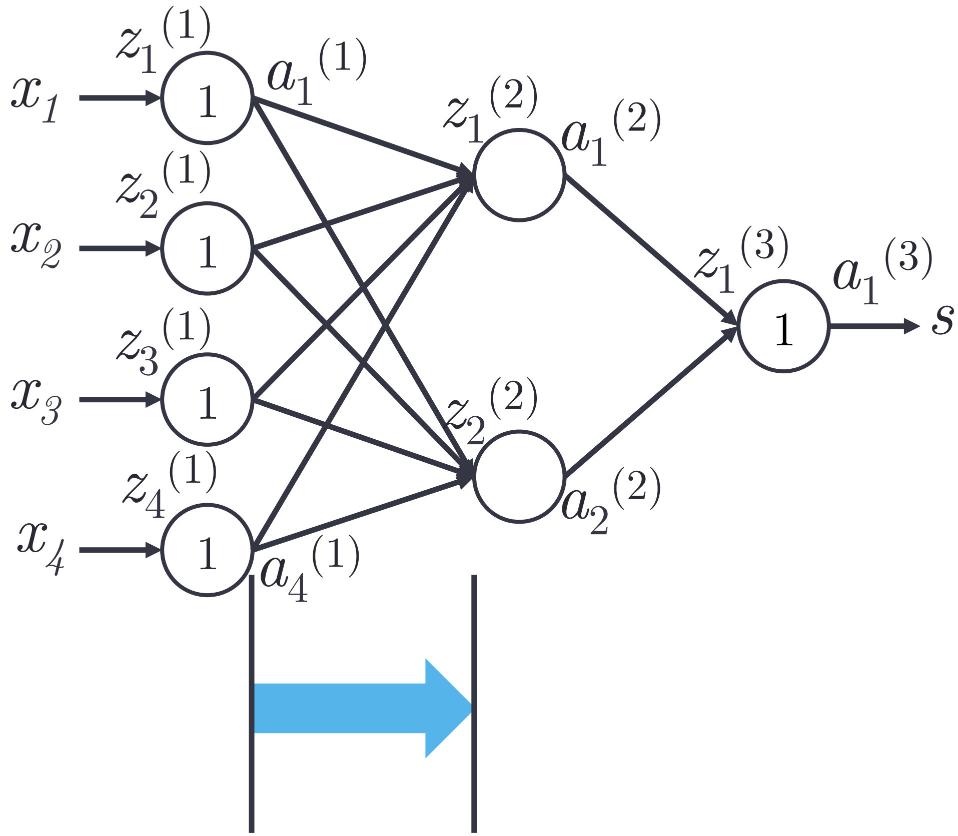
$$z_1^{(1)} = x_1$$
$$z_2^{(1)} = x_2$$
$$z_3^{(1)} = x_3$$
$$z_4^{(1)} = x_4$$

Our first example: Forward Pass



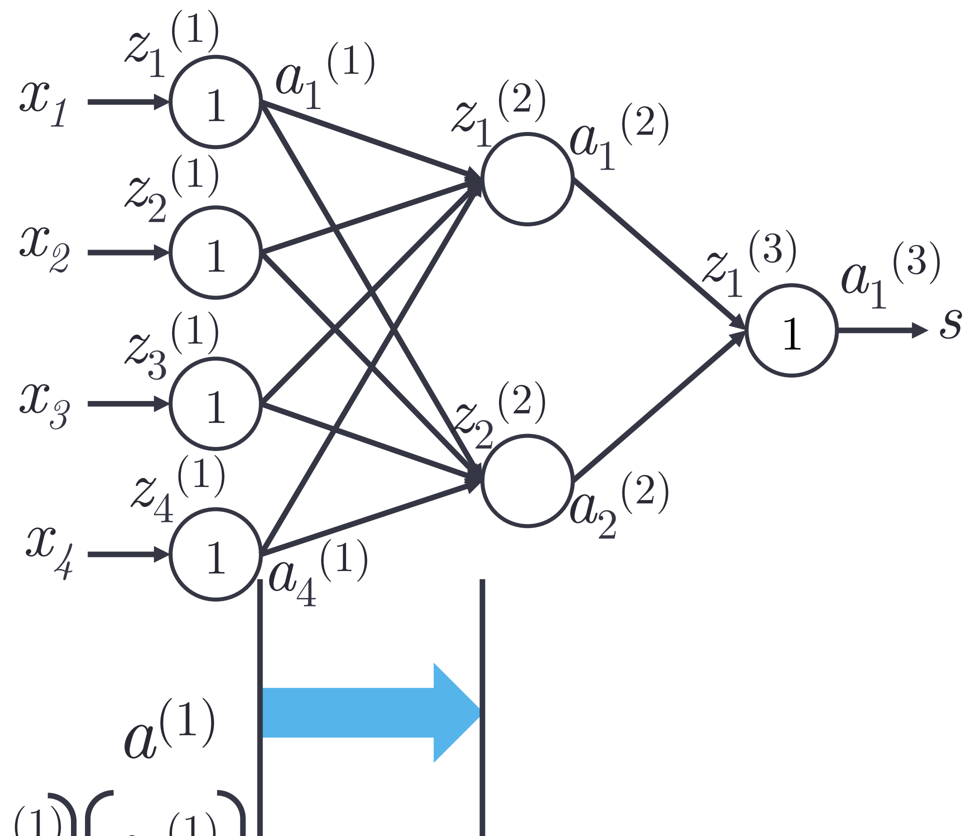
$$a_1^{(1)} = z_1^{(1)}$$
$$a_2^{(1)} = z_2^{(1)}$$
$$a_3^{(1)} = z_3^{(1)}$$
$$a_4^{(1)} = z_4^{(1)}$$

Our first example: Forward Pass



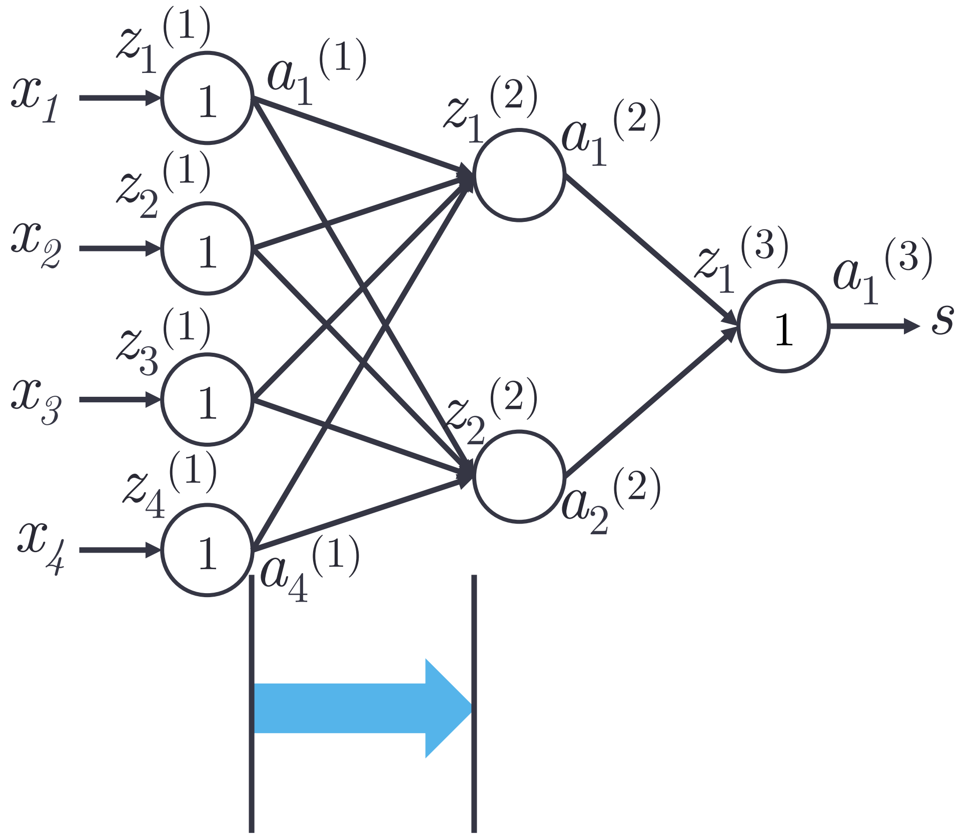
$$z_1^{(2)} = W_{11}^{(1)} a_1^{(1)} + W_{12}^{(1)} a_2^{(1)} + W_{13}^{(1)} a_3^{(1)} + W_{14}^{(1)} a_4^{(1)}$$
$$z_2^{(2)} = W_{21}^{(1)} a_1^{(1)} + W_{22}^{(1)} a_2^{(1)} + W_{23}^{(1)} a_3^{(1)} + W_{24}^{(1)} a_4^{(1)}$$

Our first example: Forward Pass



$$
\begin{array}{ccccc}
z^{(2)} & & W^{(1)} & & a^{(1)} \\
\begin{bmatrix} z_1^{(2)} \\ z_2^{(2)} \end{bmatrix} & = & \begin{bmatrix} W_{11}^{(1)} & W_{12}^{(1)} & W_{13}^{(1)} & W_{14}^{(1)} \\ W_{21}^{(1)} & W_{22}^{(1)} & W_{23}^{(1)} & W_{24}^{(1)} \end{bmatrix} & & \begin{bmatrix} a_1^{(1)} \\ a_2^{(1)} \\ a_3^{(1)} \\ a_4^{(1)} \end{bmatrix}
\end{array}
$$

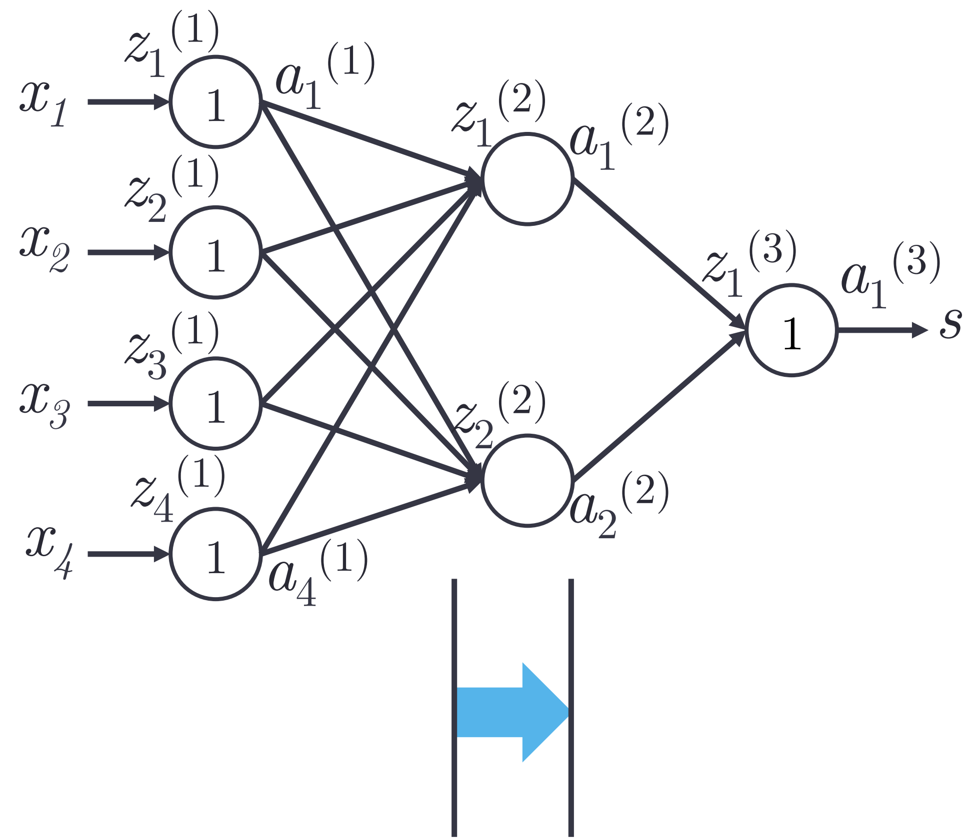
Our first example: Forward Pass



$$z^{(2)} = W^{(1)} a^{(1)}$$

Affine transformation

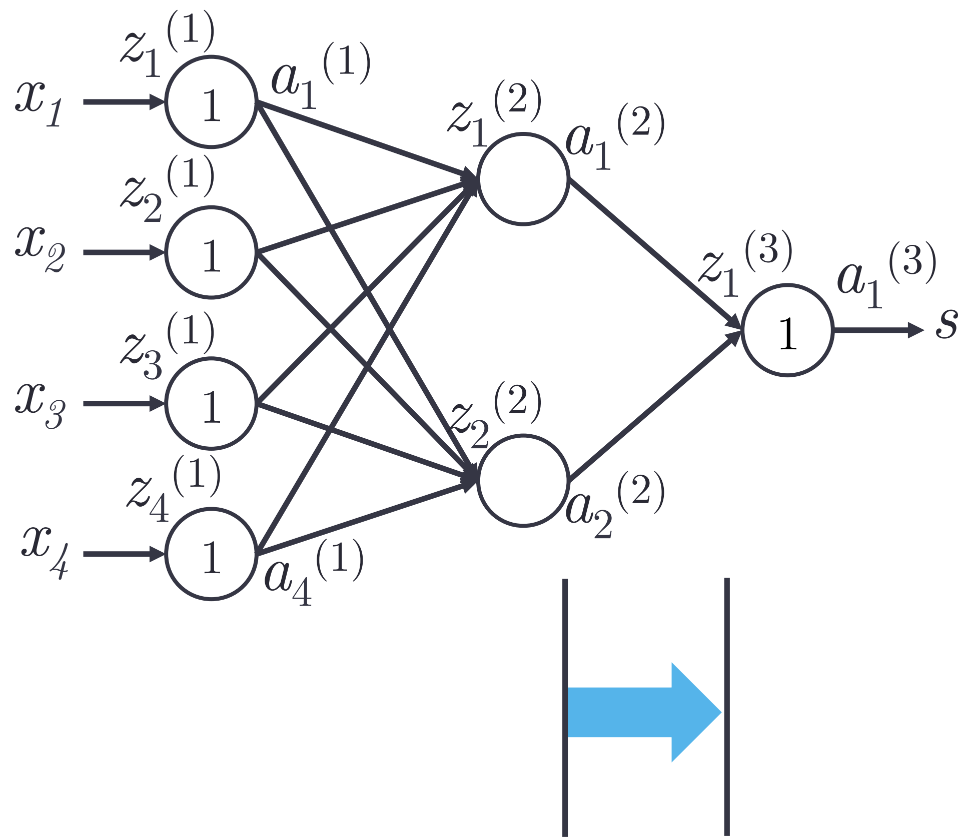
Our first example: Forward Pass



$$a^{(2)} = \sigma(z^{(2)})$$

Point-wise/Element-wise non-linearity

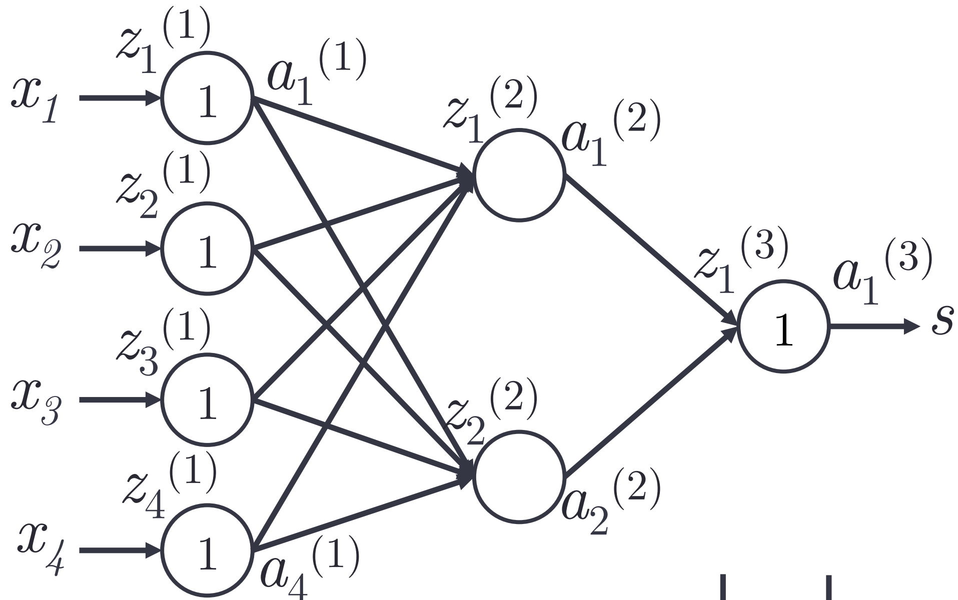
Our first example: Forward Pass

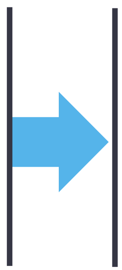


$$z^{(3)} = W^{(2)}a^{(2)}$$

Affine transformation

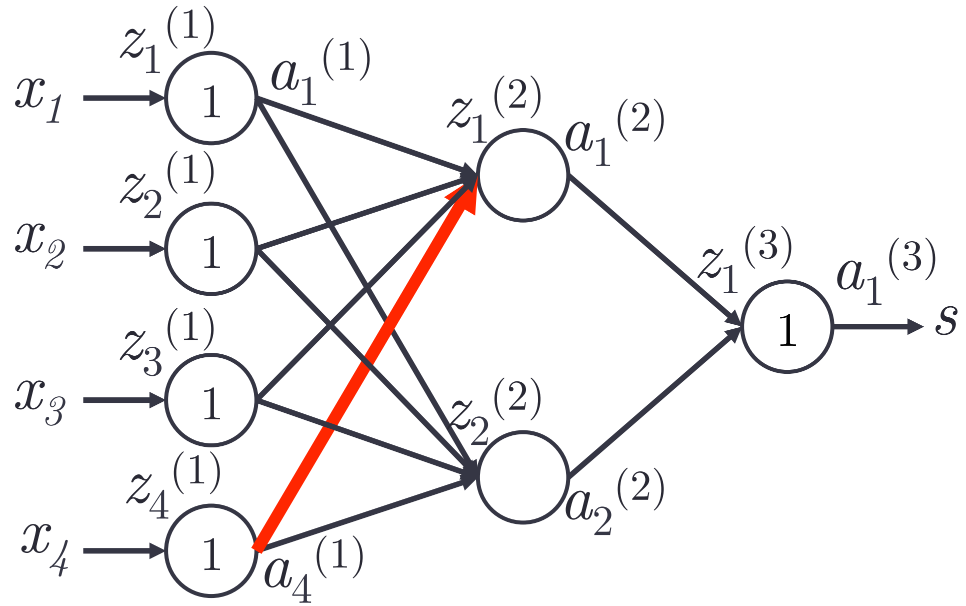
Our first example: Forward Pass





$$a^{(3)} = z^{(3)}$$
$$s = a^{(3)}$$

Our first example: Backpropagation using chain rule

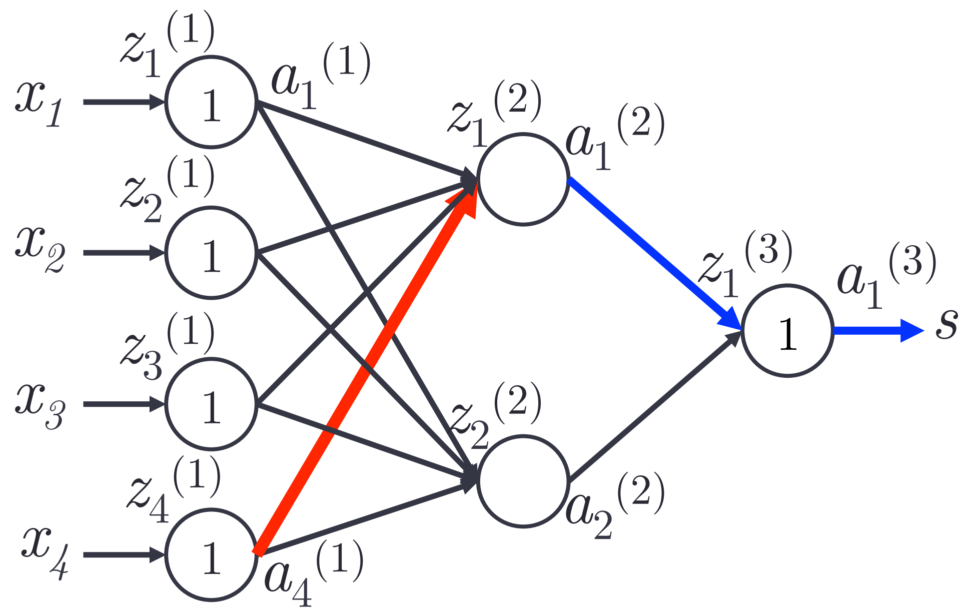


Let us try to calculate the error gradient wrt $W_{14}^{(1)}$

Thus we want to find:

$$\frac{\partial s}{\partial W_{14}^{(1)}}$$

Our first example: Backpropagation using chain rule

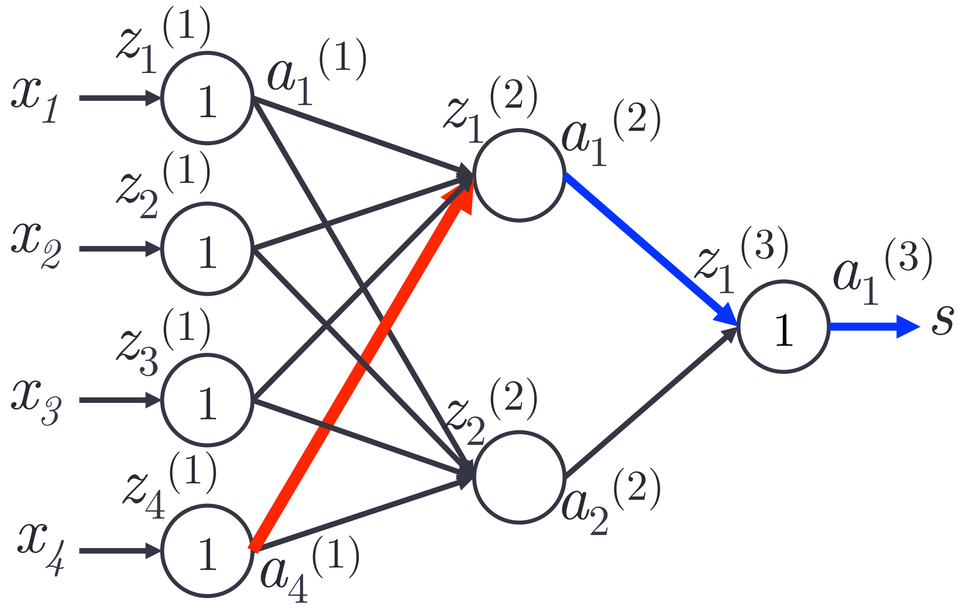


Let us try to calculate the error gradient wrt $W_{14}^{(1)}$

Thus we want to find:

$$\frac{\partial s}{\partial z_1^{(3)}} \frac{\partial z_1^{(3)}}{\partial a_1^{(2)}} \frac{\partial a_1^{(2)}}{\partial z_1^{(2)}} \frac{\partial z_1^{(2)}}{\partial W_{14}^{(1)}}$$

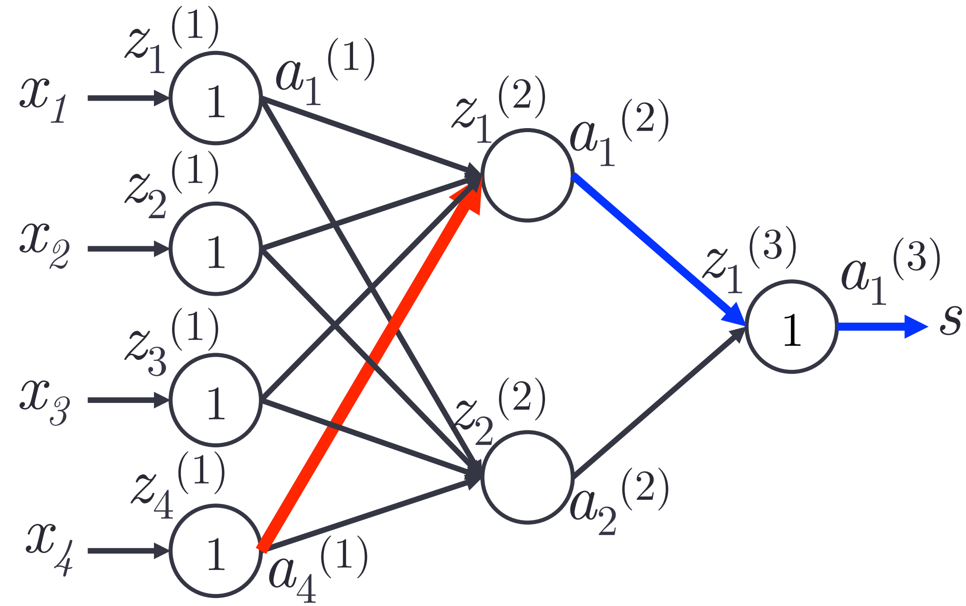
Our first example: Backpropagation using chain rule



$\boxed{\text{This}}$ is simply 1

$$\boxed{\frac{\partial s}{\partial z_1^{(3)}}} \frac{\partial z_1^{(3)}}{\partial a_1^{(2)}} \frac{\partial a_1^{(2)}}{\partial z_1^{(2)}} \frac{\partial z_1^{(2)}}{\partial W_{14}^{(1)}}$$

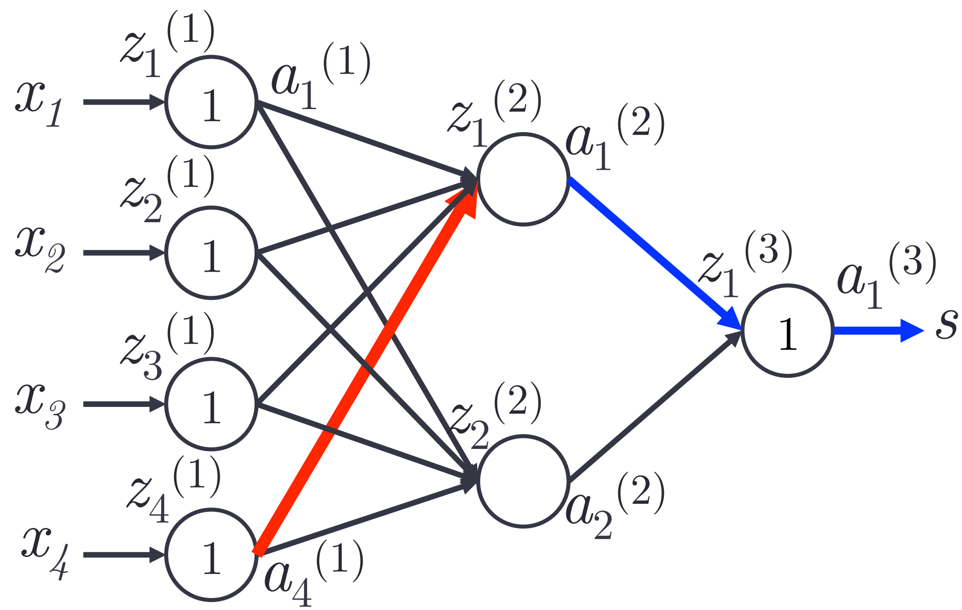
Our first example: Backpropagation using chain rule



$$\frac{\partial z_1^{(3)}}{\partial a_1^{(2)}} \frac{\partial a_1^{(2)}}{\partial z_1^{(2)}} \frac{\partial z_1^{(2)}}{\partial W_{14}^{(1)}}$$

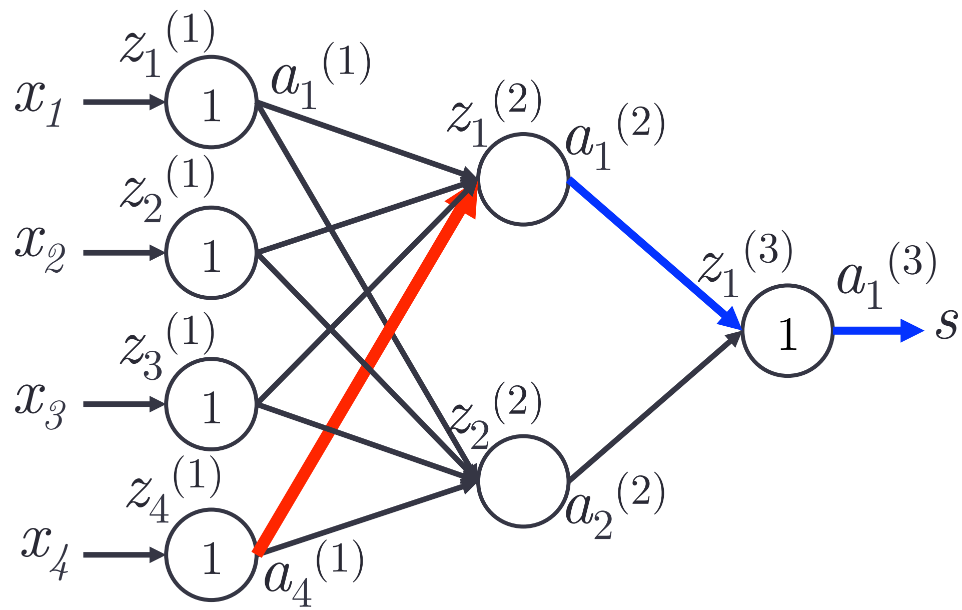
$$\frac{\partial \left(W_{11}^{(2)} a_1^{(2)} + W_{12}^{(2)} a_2^{(2)}\right)}{\partial a_1^{(2)}} \frac{\partial a_1^{(2)}}{\partial z_1^{(2)}} \frac{\partial z_1^{(2)}}{\partial W_{14}^{(1)}}$$

Our first example: Backpropagation using chain rule



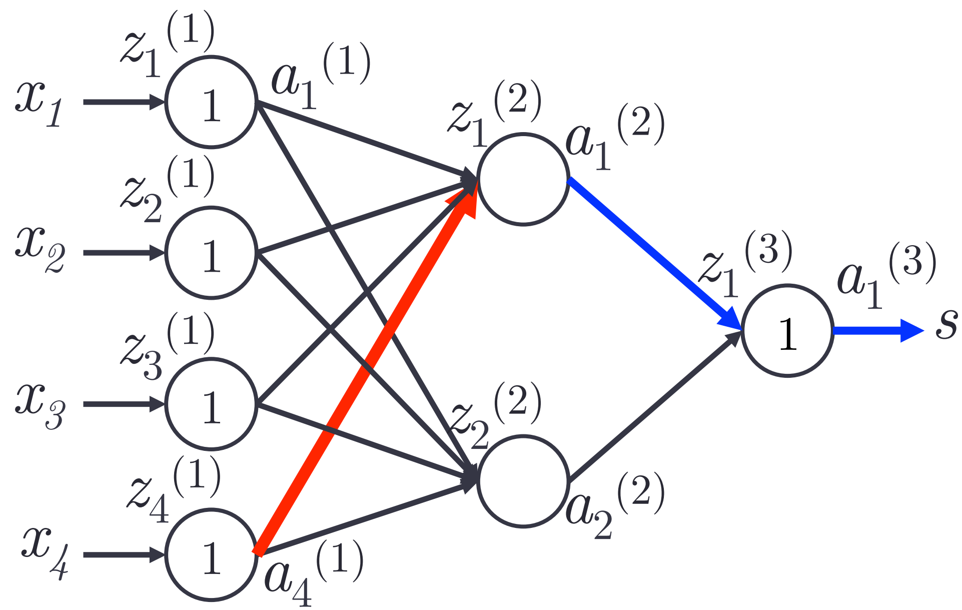
$$W_{11}^{(2)} \frac{\partial a_1^{(2)}}{\partial z_1^{(2)}} \frac{\partial z_1^{(2)}}{\partial W_{14}^{(1)}}$$

Our first example: Backpropagation using chain rule



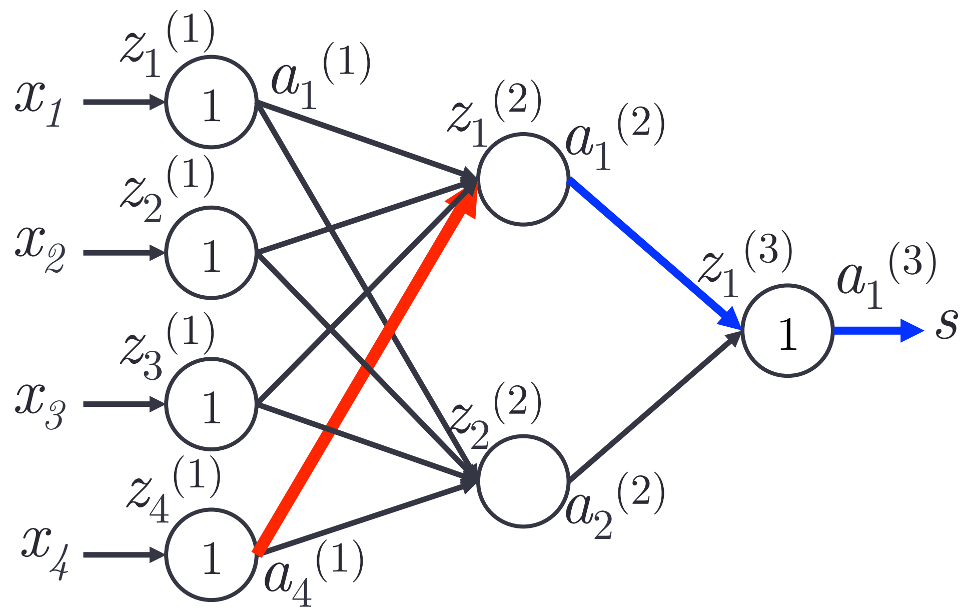
$$W_{11}^{(2)} \sigma' \left(z_1^{(2)} \right) \frac{\partial z_1^{(2)}}{\partial W_{14}^{(1)}}$$

Our first example: Backpropagation using chain rule



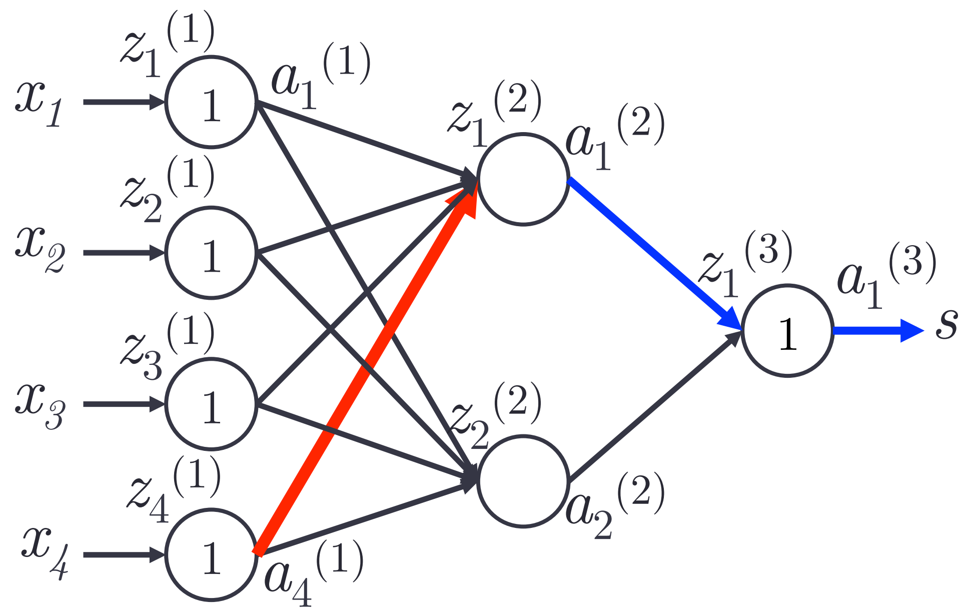
$$W_{11}^{(2)}\sigma'\left(z_1^{(2)}\right)\frac{\partial(W_{11}^{(1)}a_1^{(1)} + W_{12}^{(1)}a_2^{(1)} + W_{13}^{(1)}a_3^{(1)} + W_{14}^{(1)}a_4^{(1)})}{\partial W_{14}^{(1)}}$$

Our first example: Backpropagation using chain rule



$$\underbrace{W_{11}^{(2)}\sigma'\left(z_1^{(2)}\right)}_{\delta_1^{(2)}} a_4^{(1)}$$

Our first example: Backpropagation Observations

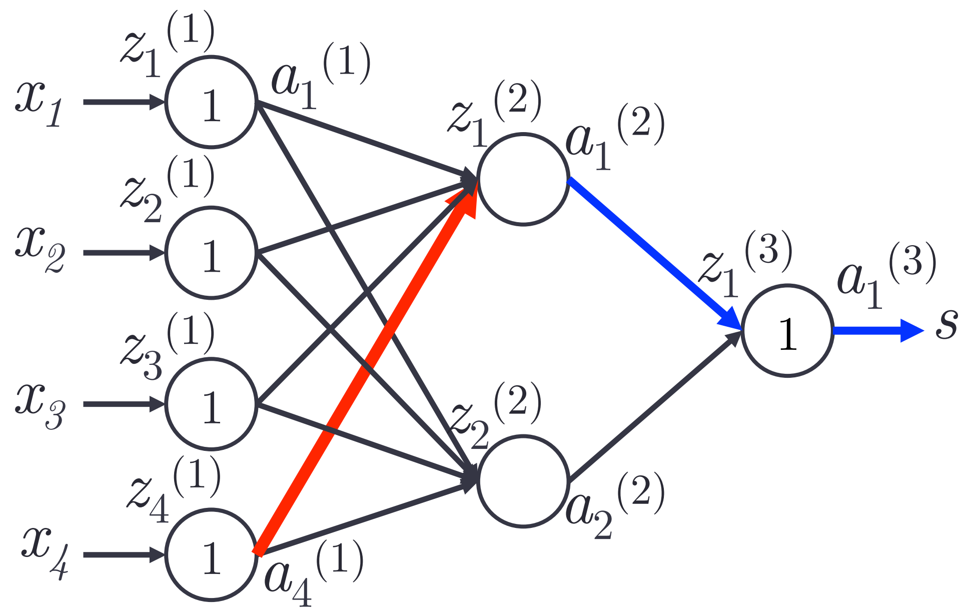


We got error gradient wrt $W_{14}^{(1)}$

Required:

- the signal forwarded by $W_{14}^{(1)} = a_4^{(1)}$
- the error propagating backwards $W_{11}^{(2)}$
- the local gradient $\sigma'(z_1^{(2)})$

Our first example: Backpropagation Observations



We tried to get error gradient wrt $W_{14}^{(1)}$

Required:

- the signal forwarded by $W_{14}^{(1)} = a_4^{(1)}$
- the error propagating backwards $W_{11}^{(2)}$
- the local gradient $\sigma'(z_1^{(2)})$

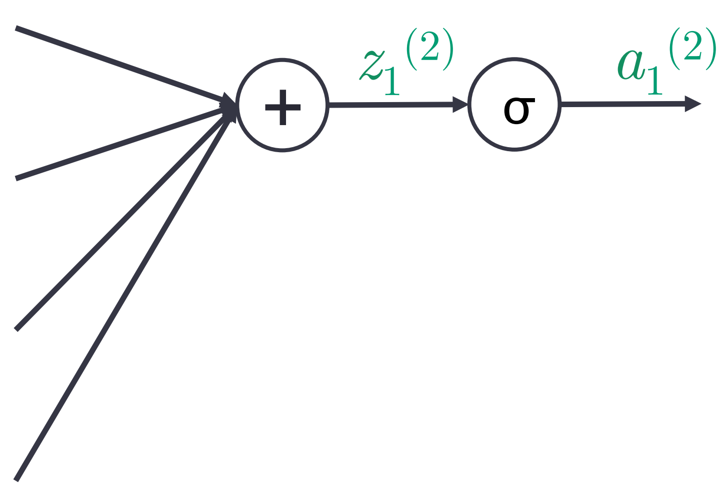
We can do this for all of $W^{(1)}$:

$$\begin{bmatrix} \delta_1^{(2)} a_1^{(1)} & \delta_1^{(2)} a_2^{(1)} & \delta_1^{(2)} a_3^{(1)} & \delta_1^{(2)} a_4^{(1)} \\ \delta_2^{(2)} a_1^{(1)} & \delta_2^{(2)} a_2^{(1)} & \delta_2^{(2)} a_3^{(1)} & \delta_2^{(2)} a_4^{(1)} \end{bmatrix}$$

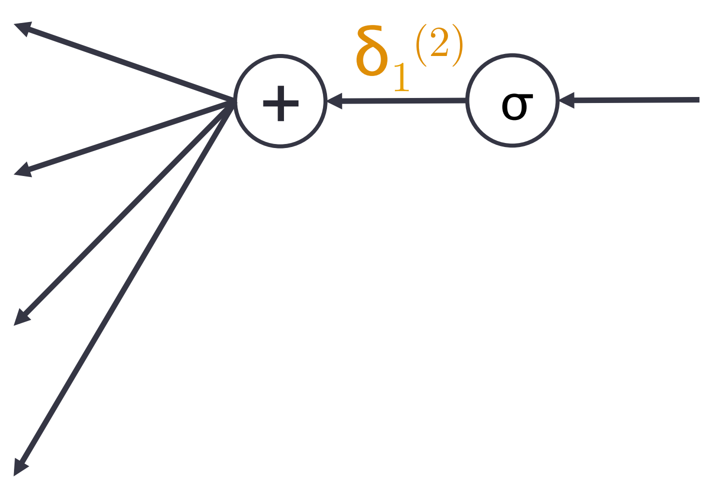
(as outer product)

$$\begin{bmatrix} \delta_1^{(2)} \\ \delta_2^{(2)} \end{bmatrix} \begin{bmatrix} a_1^{(1)} & a_2^{(1} & a_3^{(1)} & a_4^{(1)} \end{bmatrix}$$

Our first example: Let us define δ



Recall that this is forward pass



This is the backpropagation

$\delta_1^{(2)}$ is the error flowing backwards at the same point where $z_1^{(2)}$ passed forwards. Thus it is simply the gradient of the error wrt $z_1^{(2)}$.

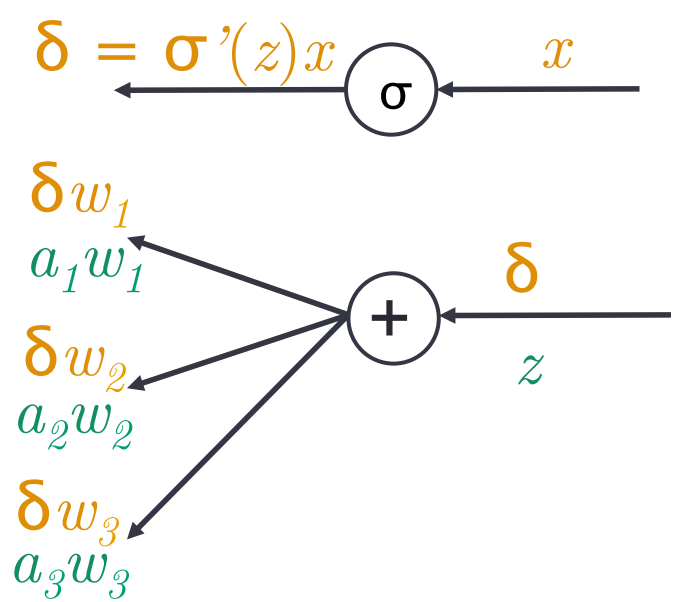
Our first example: Backpropagation using error vectors

The chain rule of differentiation just boils down very simple patterns in error backpropagation:

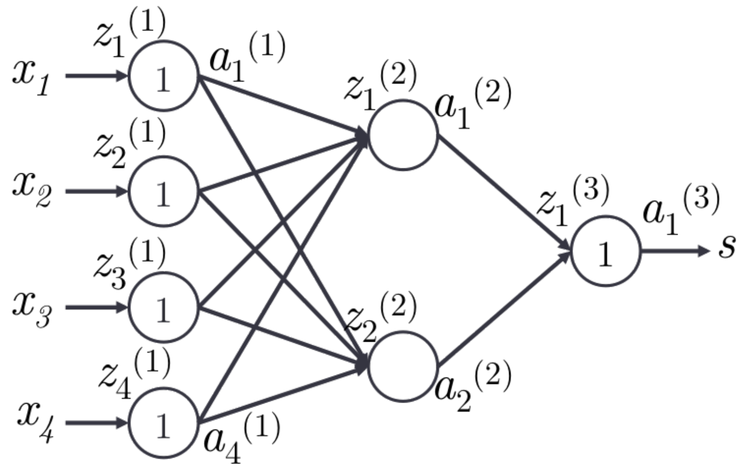
Orange = Backprop.
Green = Fwd. Pass

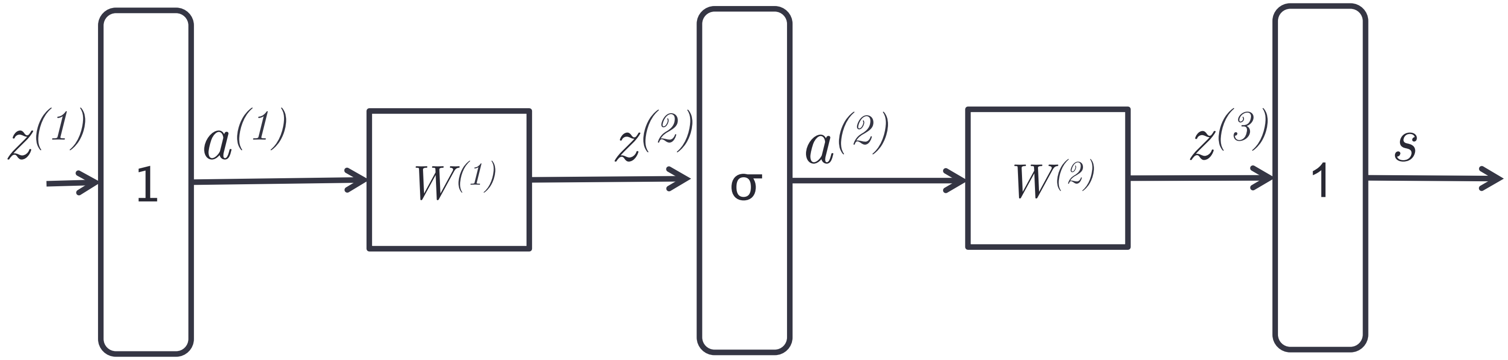
1. An error x flowing backwards passes a neuron by getting amplified by the local gradient.

2. An error δ that needs to go through an affine transformation distributes itself in the way signal combined in forward pass.

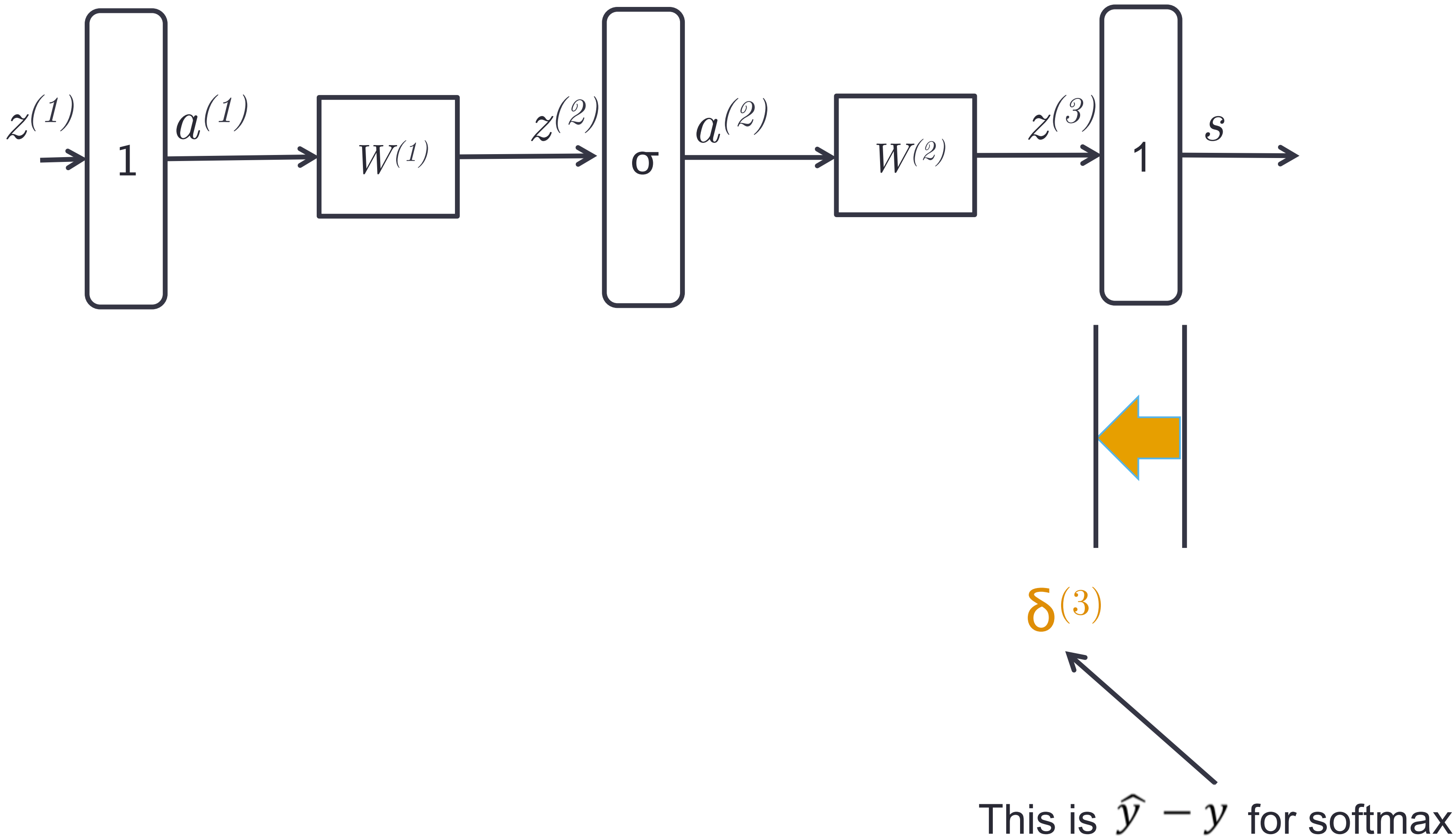


Our first example: Backpropagation using error vectors

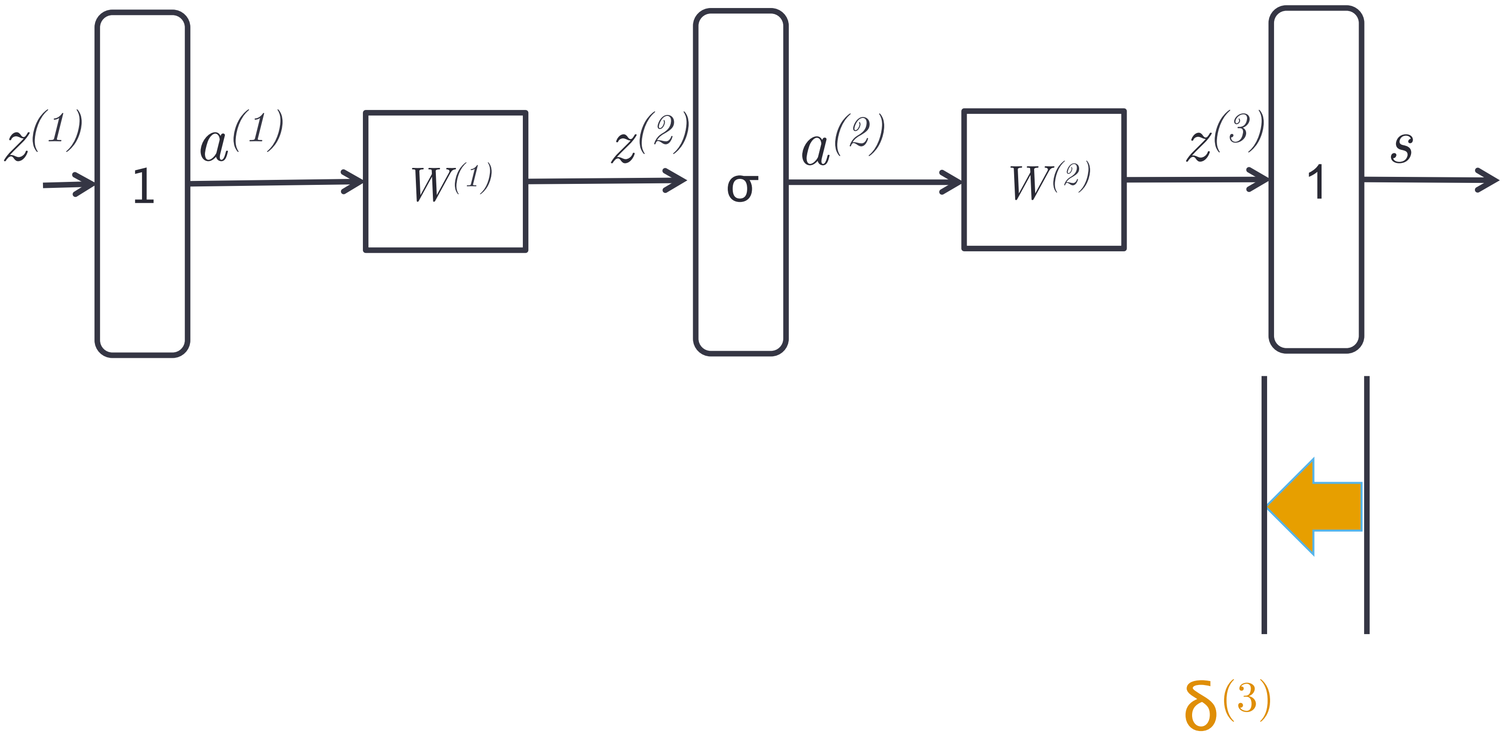




Our first example: Backpropagation using error vectors

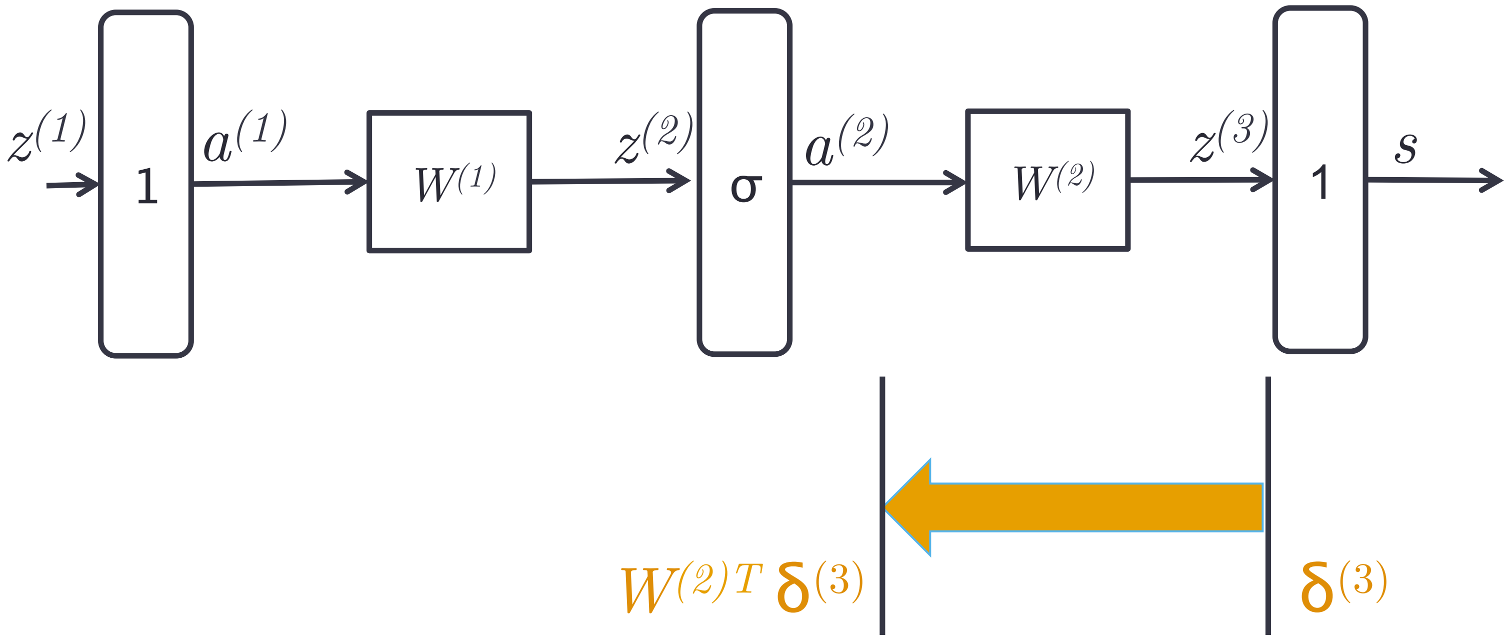


Our first example: Backpropagation using error vectors



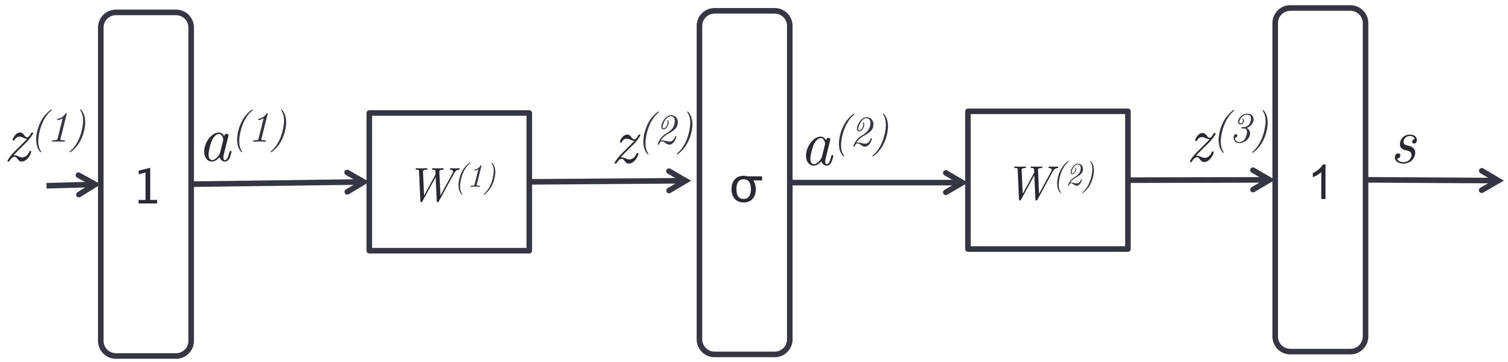
$$\text{Gradient w.r.t } W^{(2)} = \delta^{(3)} a^{(2)T}$$

Our first example: Backpropagation using error vectors

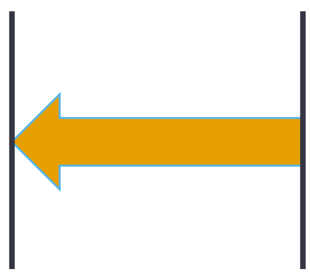


--Reusing the $\delta^{(3)}$ for downstream updates.
--Moving error vector across affine transformation simply requires multiplication with the transpose of forward matrix
--Notice that the dimensions will line up perfectly too!

Our first example: Backpropagation using error vectors



$\sigma'(z^{(2)}) \odot W^{(2)T} \delta^{(3)}$

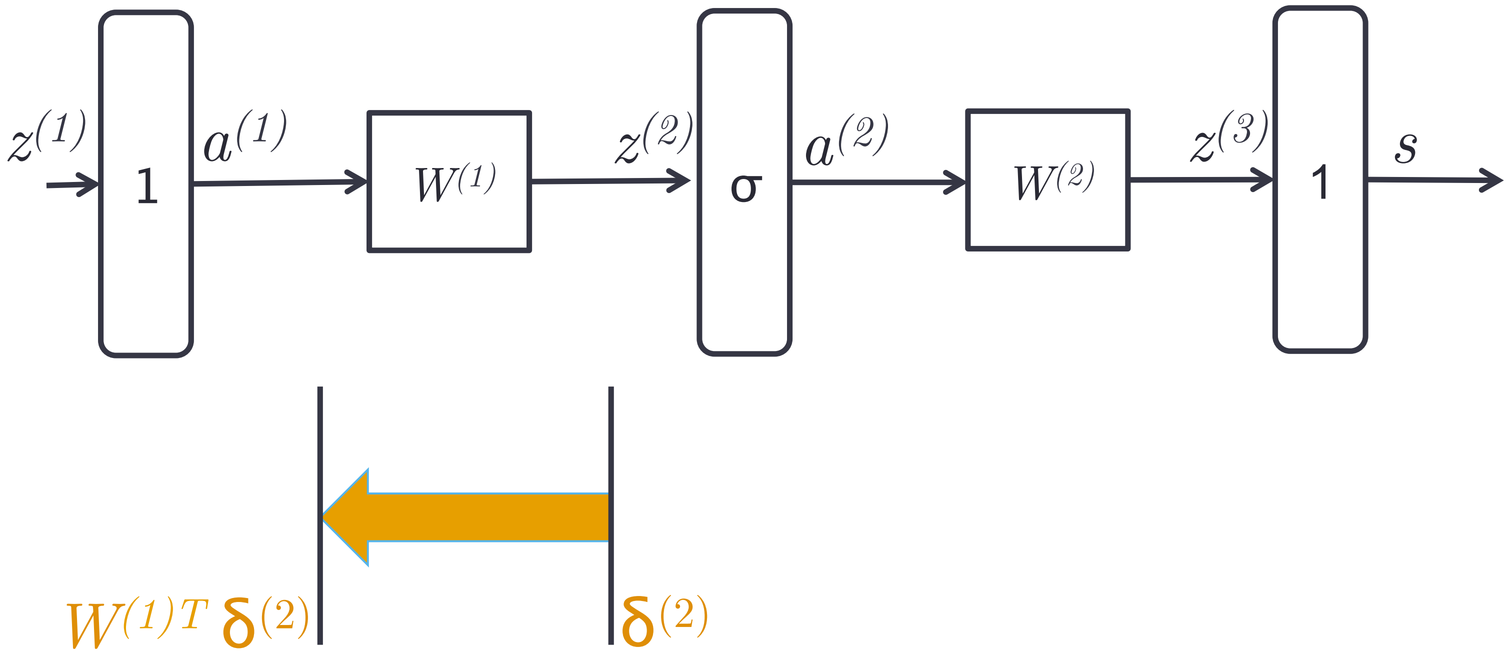


$W^{(2)T} \delta^{(3)}$

$= \delta^{(2)}$

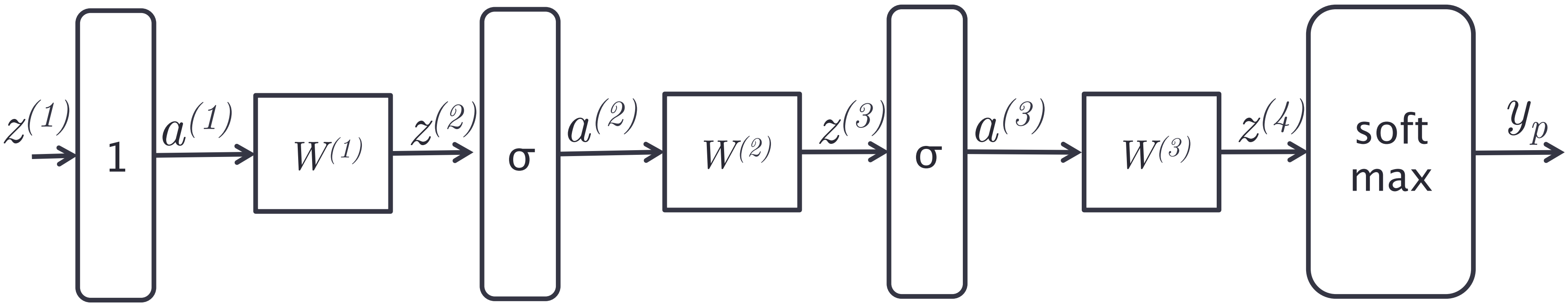
--Moving error vector across point-wise non-linearity requires point-wise multiplication with local gradient of the non-linearity

Our first example: Backpropagation using error vectors

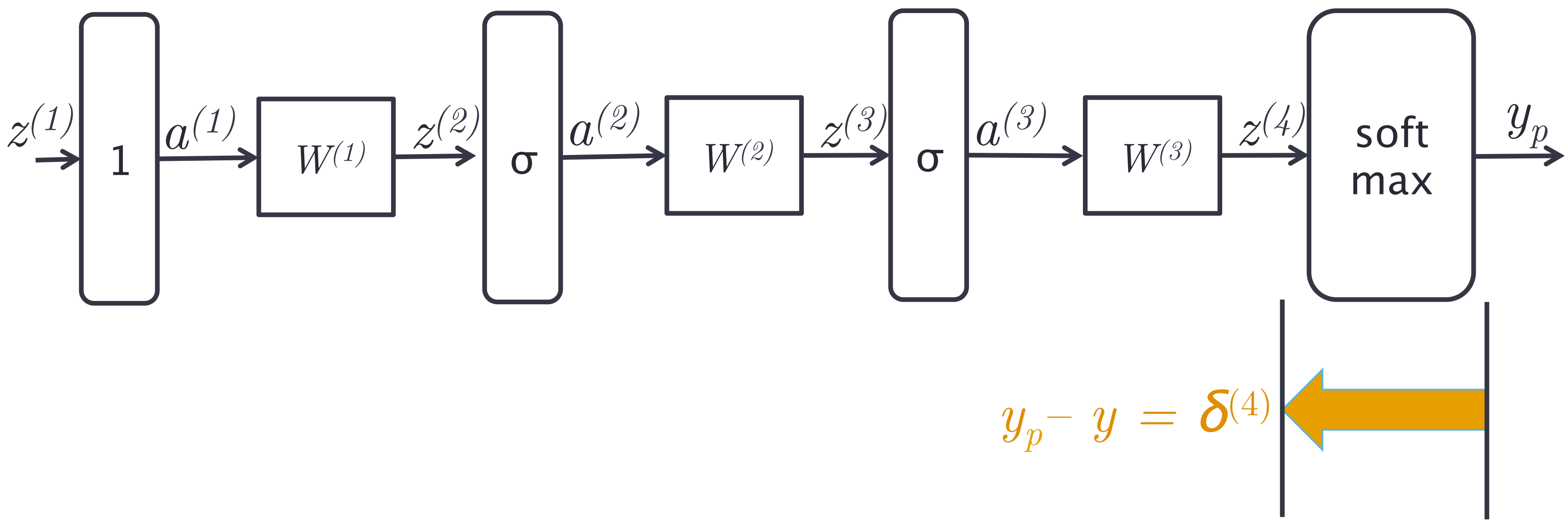


Gradient w.r.t $W^{(1)} = \delta^{(2)} a^{(1)T}$

Our second example (4-layer network): Backpropagation using error vectors

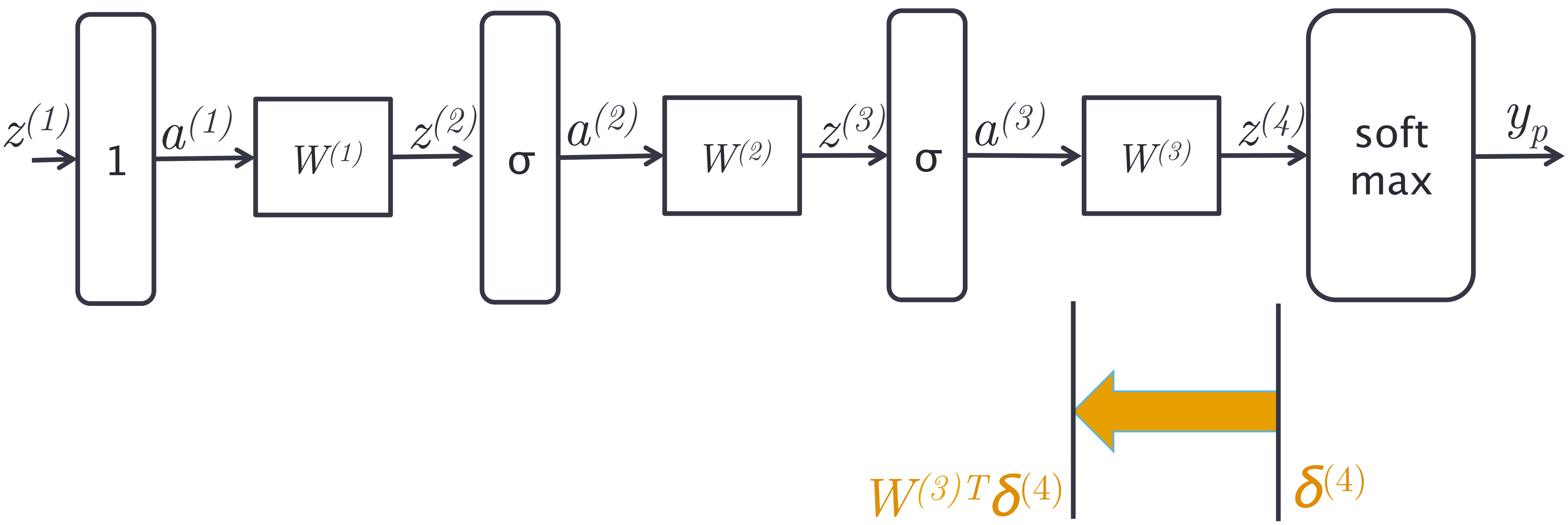


Our second example (4-layer network): Backpropagation using error vectors

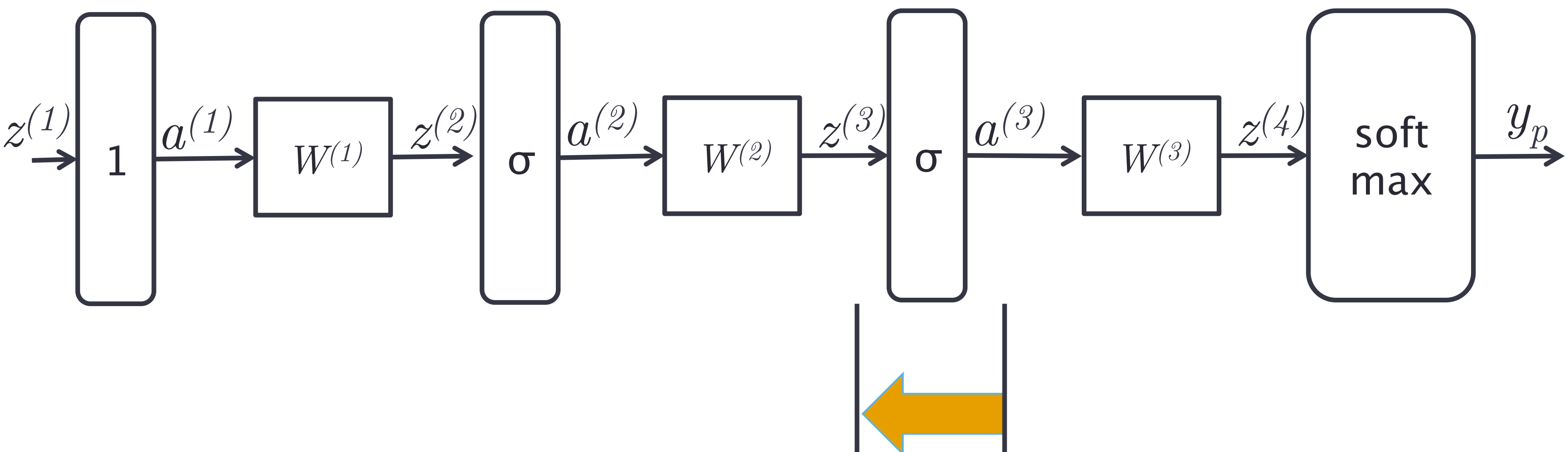


Our second example (4-layer network): Backpropagation using error vectors

$$\text{Grad } W^{(3)} = \delta^{(4)} a^{(3)T}$$



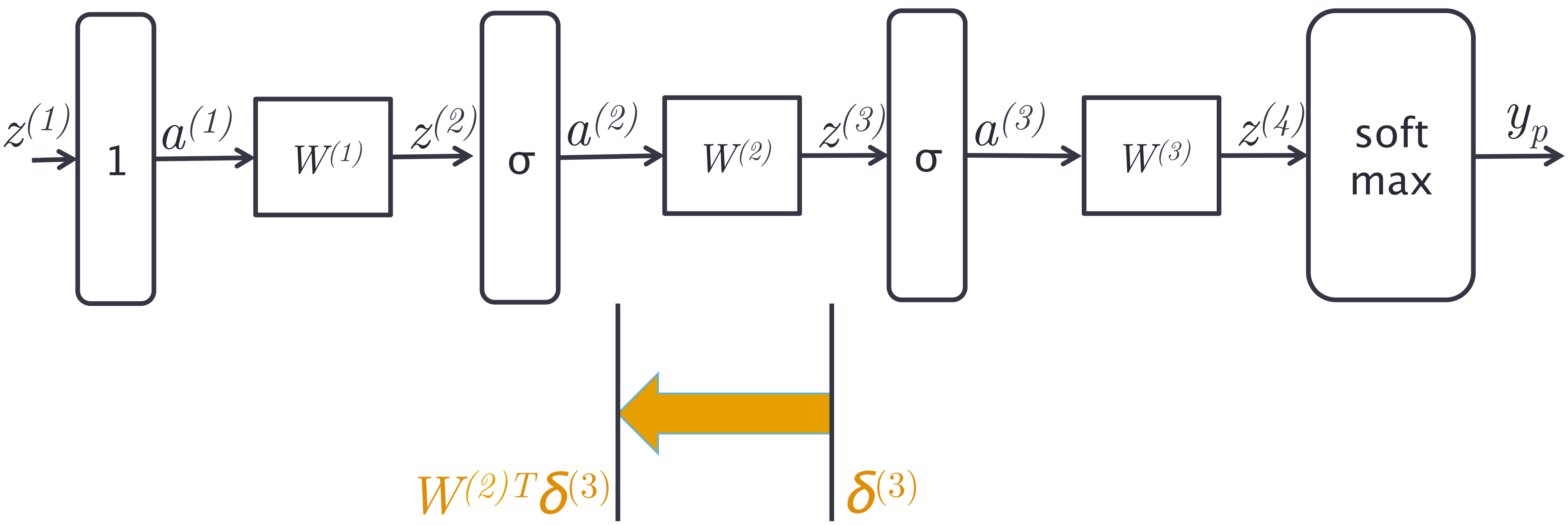
Our second example (4-layer network): Backpropagation using error vectors



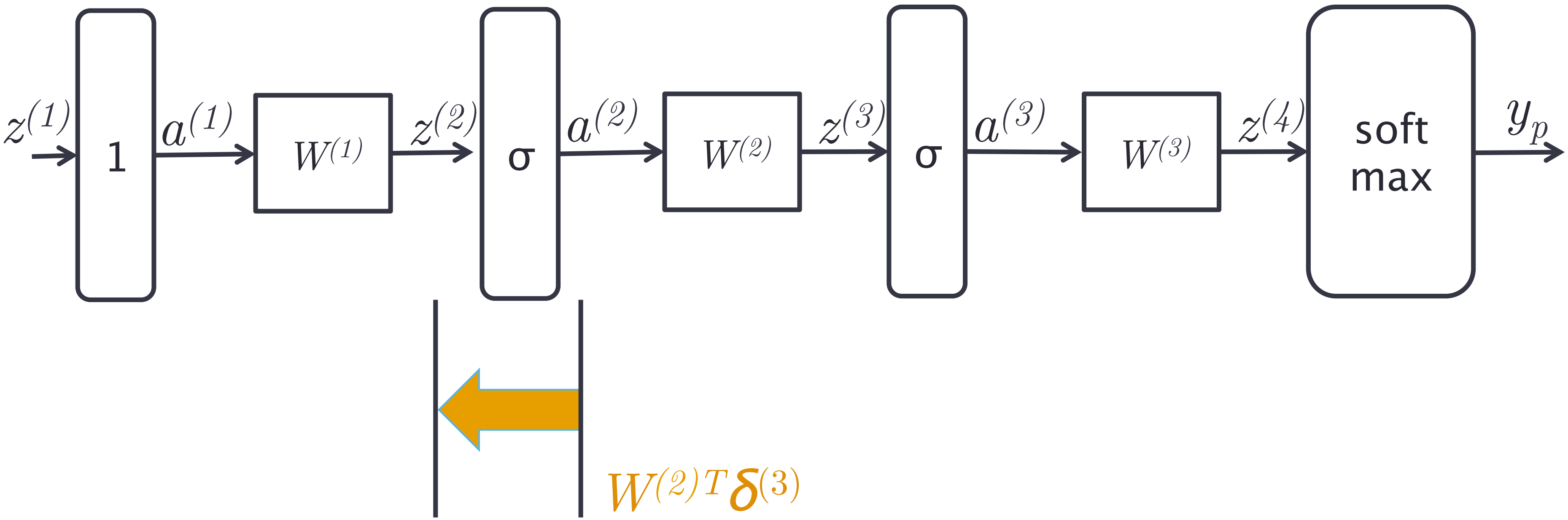
$\delta^{(3)} = \sigma'(z^{(3)}) \odot W^{(3)T}\delta^{(4)}$ $W^{(3)T}\delta^{(4)}$

Our second example (4-layer network): Backpropagation using error vectors

$$\text{Grad } W^{(2)} = \delta^{(3)} a^{(2)T}$$



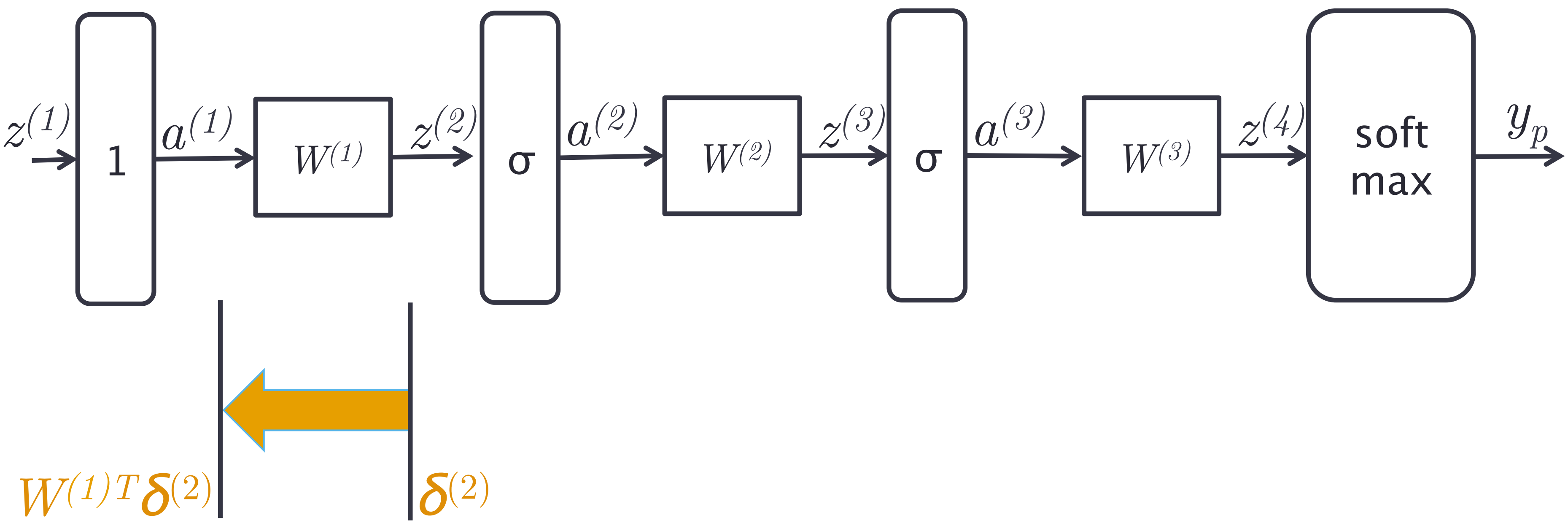
Our second example (4-layer network): Backpropagation using error vectors



$$\delta^{(2)} = \sigma'(z^{(2)}) \odot W^{(2)T}\delta^{(3)}$$

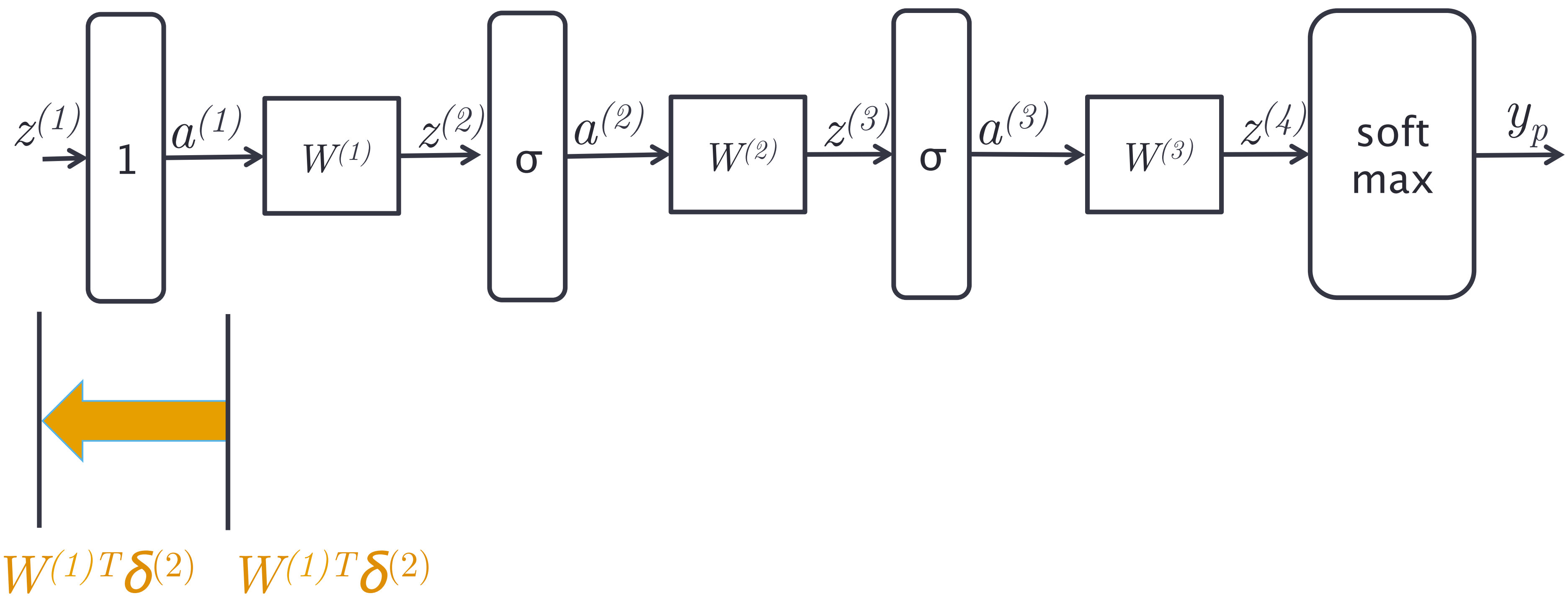
Our second example (4-layer network): Backpropagation using error vectors

$$\text{Grad } W^{(1)} = \delta^{(2)} a^{(1)T}$$



Our second example (4-layer network): Backpropagation using error vectors

$$\text{Grad wrt input vector} = W^{(1)T}\delta^{(2)}$$



CS224D Midterm Review

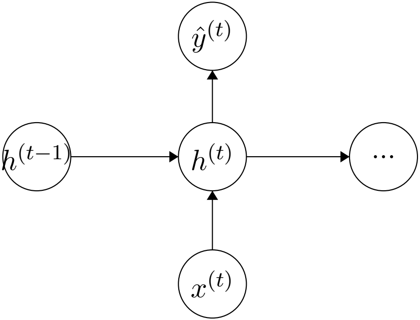
Ian Tenney

May 4, 2015

Outline

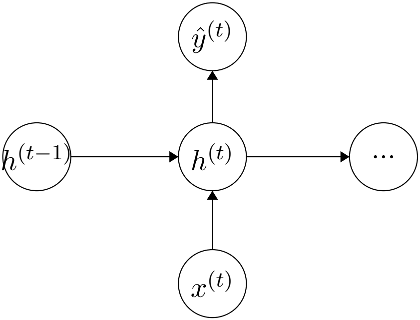
Outline

Basic RNN Structure



- Basic RNN ("Elman network")
- You've seen this on Assignment #2 (and also in Lecture #5)

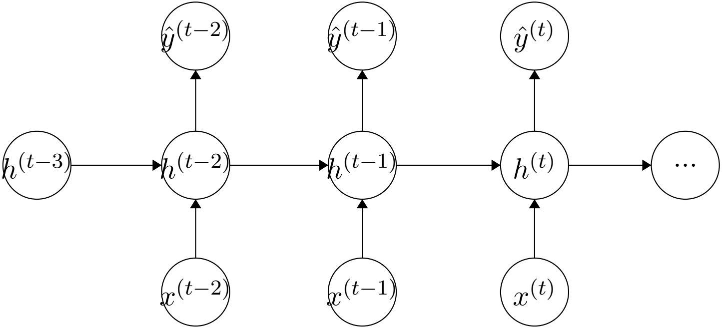
Basic RNN Structure



- Two layers between input and prediction, plus hidden state

$$
\begin{aligned}
h^{(t)} &= \text{sigmoid}\left(Hh^{(t-1)} + Wx^{(t)} + b_1\right) \\
\hat{y}^{(t)} &= \text{softmax}\left(Uh^{(t)} + b_2\right)
\end{aligned}
$$

Unrolled RNN



- Helps to think about as “unrolled” network: distinct nodes for each timestep
- Just do backprop on this! Then combine shared gradients.

Backprop on RNN

- Usual cross-entropy loss (k-class):

$$\begin{aligned}\bar{P}(y^{(t)} = j \mid x^{(t)}, \ldots, x^{(1)}) &= \hat{y}_j^{(t)} \\ J^{(t)}(\theta) &= -\sum_{j=1}^{k} y_j^{(t)} \log \hat{y}_j^{(t)}\end{aligned}$$

- Just do backprop on this! First timestep ($\tau = 1$):

$$\frac{\partial J^{(t)}}{\partial U} \qquad \frac{\partial J^{(t)}}{\partial b_2}$$

$$\left.\frac{\partial J^{(t)}}{\partial H}\right|_{(t)} \qquad \frac{\partial J^{(t)}}{\partial h^{(t)}} \qquad \left.\frac{\partial J^{(t)}}{\partial W}\right|_{(t)} \qquad \frac{\partial J^{(t)}}{\partial x^{(t)}}$$

Backprop on RNN

- First timestep ($s = 0$):

$$\frac{\partial J^{(t)}}{\partial U} \qquad \frac{\partial J^{(t)}}{\partial b_2}$$

$$\left.\frac{\partial J^{(t)}}{\partial H}\right|_{(t)} \qquad \frac{\partial J^{(t)}}{\partial h^{(t)}} \qquad \left.\frac{\partial J^{(t)}}{\partial W}\right|_{(t)} \qquad \frac{\partial J^{(t)}}{\partial x^{(t)}}$$

- Back in time ($s = 1, 2, \ldots, \tau - 1$)

$$\left.\frac{\partial J^{(t)}}{\partial H}\right|_{(t-s)} \qquad \frac{\partial J^{(t)}}{\partial h^{(t-s)}} \qquad \left.\frac{\partial J^{(t)}}{\partial W}\right|_{(t-s)} \qquad \frac{\partial J^{(t)}}{\partial x^{(t-s)}}$$

Backprop on RNN

Yuck, that's a lot of math!

- Actually, it's not so bad.
- Solution: error vectors (δ)

Making sense of the madness

- Chain rule to the rescue!
- $a^{(t)} = Uh^{(t)} + b_2$
- $\hat{y}^{(t)} = \text{softmax}(a^{(t)})$
- Gradient is *transpose* of Jacobian:

$$\nabla_a J = \left(\frac{\partial J^{(t)}}{\partial a^{(t)}}\right)^T = \hat{y}^{(t)} - y^{(t)} = \delta^{(2)(t)} \qquad \in \mathbb{R}^{k\times 1}$$

- Now dimensions work out:

$$\frac{\partial J^{(t)}}{\partial a^{(t)}} \cdot \frac{\partial a^{(t)}}{\partial b_2} = (\delta^{(2)(t)})^T I \qquad \in \mathbb{R}^{(1\times k)\cdot(k\times k)} = \mathbb{R}^{1\times k}$$

Making sense of the madness

- Chain rule to the rescue!
- $a^{(t)} = Uh^{(t)} + b_2$
- $\hat{y}^{(t)} = \text{softmax}(a^{(t)})$
- Gradient is *transpose* of Jacobian:

$$\nabla_a J = \left(\frac{\partial J^{(t)}}{\partial a^{(t)}}\right)^T = \hat{y}^{(t)} - y^{(t)} = \delta^{(2)(t)} \qquad \in \mathbb{R}^{k \times 1}$$

- Now dimensions work out:

$$\frac{\partial J^{(t)}}{\partial a^{(t)}} \cdot \frac{\partial a^{(t)}}{\partial b_2} = (\delta^{(2)(t)})^T I \qquad \in \mathbb{R}^{(1 \times k) \cdot (k \times k)} = \mathbb{R}^{1 \times k}$$

Making sense of the madness

- Chain rule to the rescue!
- $a^{(t)} = Uh^{(t)} + b_2$
- $\hat{y}^{(t)} = \text{softmax}(a^{(t)})$
- Gradient is *transpose* of Jacobian:

$$\nabla_a J = \left(\frac{\partial J^{(t)}}{\partial a^{(t)}} \right)^T = \hat{y}^{(t)} - y^{(t)} = \delta^{(2)(t)} \quad \in \mathbb{R}^{k \times 1}$$

- Now dimensions work out:

$$\frac{\partial J^{(t)}}{\partial a^{(t)}} \cdot \frac{\partial a^{(t)}}{\partial b_2} = (\delta^{(2)(t)})^T I \quad \in \mathbb{R}^{(1 \times k) \cdot (k \times k)} = \mathbb{R}^{1 \times k}$$

Making sense of the madness

- Chain rule to the rescue!
- $a^{(t)} = Uh^{(t)} + b_2$
- $\hat{y}^{(t)} = \text{softmax}(a^{(t)})$
- Matrix dimensions get weird:

$$\frac{\partial a^{(t)}}{\partial U} \quad \in \mathbb{R}^{k \times (k \times D_h)}$$

- But we don't need fancy tensors:

$$\nabla_U J^{(t)} = \left(\frac{\partial J^{(t)}}{\partial a^{(t)}} \cdot \frac{\partial a^{(t)}}{\partial U} \right)^T = \delta^{(2)(t)} (h^{(t)})^T \quad \in \mathbb{R}^{k \times D_h}$$

- NumPy: `self.grads.U += outer(d2, hs[t])`

Making sense of the madness

- Chain rule to the rescue!
- $a^{(t)} = Uh^{(t)} + b_2$
- $\hat{y}^{(t)} = \text{softmax}(a^{(t)})$
- Matrix dimensions get weird:

$$\frac{\partial a^{(t)}}{\partial U} \quad \in \mathbb{R}^{k \times (k \times D_h)}$$

- But we don't need fancy tensors:

$$\nabla_U J^{(t)} = \left(\frac{\partial J^{(t)}}{\partial a^{(t)}} \cdot \frac{\partial a^{(t)}}{\partial U} \right)^T = \delta^{(2)(t)} (h^{(t)})^T \quad \in \mathbb{R}^{k \times D_h}$$

- NumPy: `self.grads.U += outer(d2, hs[t])`

Going deeper

- Really just need one simple pattern:
- $z^{(t)} = Hh^{(t-1)} + Wx^{(t)} + b_1$
- $h^{(t)} = f(z^{(t)})$
- Compute error delta ($s = 0, 1, 2, \ldots$):
  - From top: $\delta^{(t)} = \left[h^{(t)} \circ (1 - h^{(t)})\right] \circ U^T \delta^{(2)(t)}$
  - Deeper: $\delta^{(t-s)} = \left[h^{(t-s)} \circ (1 - h^{(t-s)})\right] \circ H^T \delta^{(t-s+1)}$
- These are just chain-rule expansions!

$$\frac{\partial J^{(t)}}{\partial z^{(t)}} = \frac{\partial J^{(t)}}{\partial a^{(t)}} \cdot \frac{\partial a^{(t)}}{\partial h^{(t)}} \cdot \frac{\partial h^{(t)}}{\partial z^{(t)}} = (\delta^{(t)})^T$$

Going deeper

- Really just need one simple pattern:
- $z^{(t)} = Hh^{(t-1)} + Wx^{(t)} + b_1$
- $h^{(t)} = f(z^{(t)})$
- Compute error delta ($s = 0, 1, 2, \ldots$):
  - From top: $\delta^{(t)} = \left[h^{(t)} \circ (1 - h^{(t)})\right] \circ U^T \delta^{(2)(t)}$
  - Deeper: $\delta^{(t-s)} = \left[h^{(t-s)} \circ (1 - h^{(t-s)})\right] \circ H^T \delta^{(t-s+1)}$
- These are just chain-rule expansions!

$$\frac{\partial J^{(t)}}{\partial z^{(t)}} = \frac{\partial J^{(t)}}{\partial a^{(t)}} \cdot \frac{\partial a^{(t)}}{\partial h^{(t)}} \cdot \frac{\partial h^{(t)}}{\partial z^{(t)}} = (\delta^{(t)})^T$$

Going deeper

- These are just chain-rule expansions!

$$\left.\frac{\partial J^{(t)}}{\partial b_1}\right|_{(t)} = \left(\frac{\partial J^{(t)}}{\partial a^{(t)}} \cdot \frac{\partial a^{(t)}}{\partial h^{(t)}} \cdot \frac{\partial h^{(t)}}{\partial z^{(t)}}\right) \cdot \frac{\partial z^{(t)}}{\partial b_1} = (\delta^{(t)})^T \frac{\partial z^{(t)}}{\partial b_1}$$

$$\left.\frac{\partial J^{(t)}}{\partial H}\right|_{(t)} = \left(\frac{\partial J^{(t)}}{\partial a^{(t)}} \cdot \frac{\partial a^{(t)}}{\partial h^{(t)}} \cdot \frac{\partial h^{(t)}}{\partial z^{(t)}}\right) \cdot \frac{\partial z^{(t)}}{\partial H} = (\delta^{(t)})^T \frac{\partial z^{(t)}}{\partial H}$$

$$\frac{\partial J^{(t)}}{\partial z^{(t-1)}} = \left(\frac{\partial J^{(t)}}{\partial a^{(t)}} \cdot \frac{\partial a^{(t)}}{\partial h^{(t)}} \cdot \frac{\partial h^{(t)}}{\partial z^{(t)}}\right) \cdot \frac{\partial z^{(t)}}{\partial h^{(t-1)}} = (\delta^{(t)})^T \frac{\partial z^{(t)}}{\partial z^{(t-1)}}$$

Going deeper

- And there's shortcuts for them too:

$$\left(\left. \frac{\partial J^{(t)}}{\partial b_1} \right|_{(t)} \right)^T = \delta^{(t)}$$

$$\left(\left. \frac{\partial J^{(t)}}{\partial H} \right|_{(t)} \right)^T = \delta^{(t)} \cdot (h^{(t-1)})^T$$

$$\left(\frac{\partial J^{(t)}}{\partial z^{(t-1)}} \right)^T = \left[h^{(t-1)} \circ (1 - h^{(t-1)}) \right] \circ H^T \delta^{(t)} = \delta^{(t-1)}$$

Outline

Motivation

- Gated units with "reset" and "output" gates
- Reduce problems with vanishing gradients

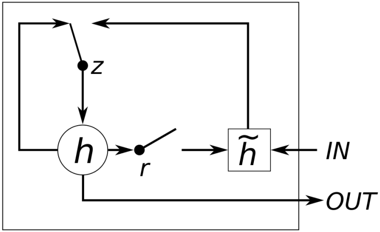


Figure : *You are likely to be eaten by a GRU.* (Figure from Chung, et al. 2014)

Intuition

- Gates z_i and r_i for *each* hidden layer neuron
- $z_i, r_i \in [0, 1]$
- $\tilde{h}$ as "candidate" hidden layer
- $\tilde{h}$, z, r all depend on on $x^{(t)}$, $h^{(t-1)}$
- $h^{(t)}$ depends on $h^{(t-1)}$ mixed with $\tilde{h}^{(t)}$

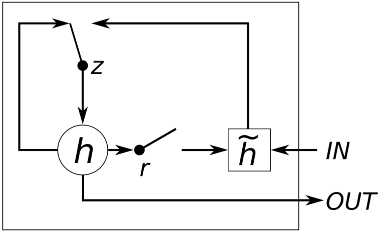


Figure : *You are likely to be eaten by a GRU.* (Figure from Chung, et al. 2014)

Equations

- $z^{(t)} = \sigma\left(W_z x^{(t)} + U_z h^{(t-1)}\right)$
- $r^{(t)} = \sigma\left(W_r x^{(t)} + U_r h^{(t-1)}\right)$
- $\tilde{h}^{(t)} = \tanh\left(W x^{(t)} + r^{(t)} \circ U h^{(t-1)}\right)$
- $h^{(t)} = z^{(t)} \circ h^{(t-1)} + (1 - z^{(t)}) \circ \tilde{h}^{(t)}$
- Optionally can have biases; omitted for clarity.

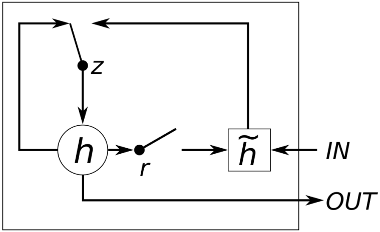


Figure : *You are likely to be eaten by a GRU.* (Figure from Chung, et al. 2014)

Same eqs. as Lecture 8, subscripts/superscripts as in Assignment #2.

Backpropagation

Multi-path to compute $\frac{\partial J}{\partial x^{(t)}}$

- Start with $\delta^{(t)} = \left(\frac{\partial J}{\partial h^{(t)}}\right)^T \quad \in \mathbb{R}^d$
- $h^{(t)} = z^{(t)} \circ h^{(t-1)} + (1 - z^{(t)}) \circ \tilde{h}^{(t)}$
- Expand chain rule into sum (*a.k.a. product rule*):

$$\begin{aligned}
\frac{\partial J}{\partial x^{(t)}} &= \frac{\partial J}{\partial h^{(t)}} \cdot \left[z^{(t)} \circ \frac{\partial h^{(t-1)}}{\partial x^{(t)}} + \frac{\partial z^{(t)}}{\partial x^{(t)}} \circ h^{(t-1)} \right] \\
&+ \frac{\partial J}{\partial h^{(t)}} \cdot \left[(1 - z^{(t)}) \circ \frac{\partial \tilde{h}^{(t)}}{\partial x^{(t)}} + \frac{\partial (1 - z^{(t)})}{\partial x^{(t)}} \circ \tilde{h}^{(t)} \right]
\end{aligned}$$

It gets (a little) better

Multi-path to compute $\frac{\partial J}{\partial x^{(t)}}$

- Drop terms that don't depend on $x^{(t)}$:

$$
\begin{aligned}
\frac{\partial J}{\partial x^{(t)}} &= \frac{\partial J}{\partial h^{(t)}} \cdot \left[z^{(t)} \circ \frac{\partial h^{(t-1)}}{\partial x^{(t)}} + \frac{\partial z^{(t)}}{\partial x^{(t)}} \circ h^{(t-1)} \right] \\
&\quad + \frac{\partial J}{\partial h^{(t)}} \cdot \left[(1 - z^{(t)}) \circ \frac{\partial \tilde{h}^{(t)}}{\partial x^{(t)}} + \frac{\partial (1 - z^{(t)})}{\partial x^{(t)}} \circ \tilde{h}^{(t)} \right] \\
&= \frac{\partial J}{\partial h^{(t)}} \cdot \left[\frac{\partial z^{(t)}}{\partial x^{(t)}} \circ h^{(t-1)} + (1 - z^{(t)}) \circ \frac{\partial \tilde{h}^{(t)}}{\partial x^{(t)}} \right] \\
&\quad - \frac{\partial J}{\partial h^{(t)}} \frac{\partial z^{(t)}}{\partial x^{(t)}} \circ \tilde{h}^{(t)}
\end{aligned}
$$

Almost there!

Multi-path to compute $\frac{\partial J}{\partial x^{(t)}}$

- Now we really just need to compute two things:
- Output gate:

$$\frac{\partial z^{(t)}}{\partial x^{(t)}} = z^{(t)} \circ (1 - z^{(t)}) \circ W_z$$

- Candidate $\tilde{h}$:

$$\begin{aligned}\frac{\partial \tilde{h}^{(t)}}{\partial x^{(t)}} = & \quad (1 - (\tilde{h}^{(t)})^2) \circ W \\ & + \quad (1 - (\tilde{h}^{(t)})^2) \circ \frac{\partial r^{(t)}}{\partial x^{(t)}} \circ U h^{(t-1)}\end{aligned}$$

- Ok, I lied - there's a third.
- Don't forget to check all paths!

Almost there!

Multi-path to compute $\frac{\partial J}{\partial x^{(t)}}$

- Now we really just need to compute two things:
- Output gate:

$$\frac{\partial z^{(t)}}{\partial x^{(t)}} = z^{(t)} \circ (1 - z^{(t)}) \circ W_z$$

- Candidate $\tilde{h}$:

$$\begin{aligned}\frac{\partial \tilde{h}^{(t)}}{\partial x^{(t)}} = & \quad (1 - (\tilde{h}^{(t)})^2) \circ W \\ & + \quad (1 - (\tilde{h}^{(t)})^2) \circ \frac{\partial r^{(t)}}{\partial x^{(t)}} \circ U h^{(t-1)}\end{aligned}$$

- Ok, I lied - there's a third.
- Don't forget to check all paths!

Almost there!

Multi-path to compute $\frac{\partial J}{\partial x^{(t)}}$

- Now we really just need to compute two things:
- Output gate:

$$\frac{\partial z^{(t)}}{\partial x^{(t)}} = z^{(t)} \circ (1 - z^{(t)}) \circ W_z$$

- Candidate $\tilde{h}$:

$$\begin{aligned}\frac{\partial \tilde{h}^{(t)}}{\partial x^{(t)}} = &\quad (1 - (\tilde{h}^{(t)})^2) \circ W \\ + &\quad (1 - (\tilde{h}^{(t)})^2) \circ \frac{\partial r^{(t)}}{\partial x^{(t)}} \circ U h^{(t-1)}\end{aligned}$$

- Ok, I lied - there's a third.
- Don't forget to check all paths!

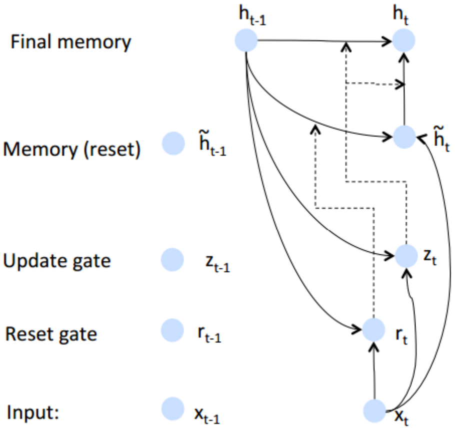
Almost there!

Multi-path to compute $\frac{\partial J}{\partial x^{(t)}}$

- Last one:
$$\frac{\partial r^{(t)}}{\partial x^{(t)}} = r^{(t)} \circ (1 - r^{(t)}) \circ W_r$$
- Now we can just add things up!
- (I'll spare you the pain...)

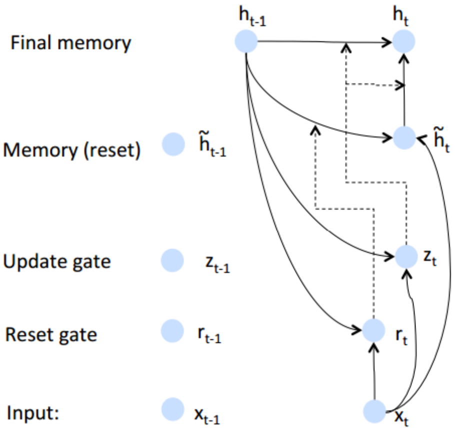
Whew.

- Why three derivatives?
- Three arrows from $x^{(t)}$ to distinct nodes
- Four paths total ($\frac{\partial z^{(t)}}{\partial x^{(t)}}$ appears twice)



Whew.

- GRUs are complicated
- All the pieces are simple
- Same matrix gradients that you've seen before



Summary

- Check your dimensions!
- Write error vectors δ; just parentheses around chain rule
- Combine simple operations to make complex network
  - Matrix-vector product
  - Activation functions (tanh, sigmoid, softmax)